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Central Spline Algorithms in the Hilbert and Frechet Spaces of Orbits

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Dedicated to the bright memory of our parents
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Ekaterine and Carlo Ugulava

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Introduction

This monograph presents the theoretical foundations of computational mathematics in Frechet spaces, which are the closest generalization of Banach spaces. A Frechet space is complete metrizable locally convex space (LCS). The monograph presents a unified approach to studying of computational processes in Banach and Frechet spaces: the study of computational processes considered in Frechet space is based on the study of computational processes in Banach spaces. Vice versa, for the study of computational processes in Banach spaces, we use the theory of Frechet spaces. In particular, attempts at a unified approach are presented in the study of the theories of best approximation, projective processes, spline algorithms, and homomorphisms, when the case of Banach spaces is part of the general approach. Many topological linear spaces of continuous, differentiable or holomorphic functions which arise in connection with various problems in analysis and its applications require a countable number of deviation measurements with respect to seminorms, which produce Frechet spaces. Therefore, it is a rapidly developing part of functional analysis and has the important applications in mathematical physics and quantum mechanics.

Mathematically, this is expressed by representing the Frechet space as the projective limit of the sequence of (local) Banach spaces associated with these seminorms, in which these Banach spaces are connected by various types of reflections (canonical mappings). These reflections give rise to the great diversity of Frechet spaces, which ultimately determine their topological, geometrical, and quantitative properties related to numerical computations.

Computational mathematics in Frechet spaces is the theory of countable measurements theory with respect to countable seminorms. In the case of a Banach space, this measurement is the only one, and it is done by means of the norm. If we approximate the element of the Frechet space, then it is necessary to calculate a countable number of approximations with respect to the seminorms, which is impossible in reality. Therefore, we approximate this element with respect to the metric obtained by metrizing the topology generated by the sequence of seminorms in Frechet space. By metrizing the space, it is possible to reduce countable measurements to a single measurement with respect to the quasinorm of the metric.

Therefore, it is natural that the results of these measurements to be essentially related to each other. In this case, it is desirable that the best approximation (as well as the ε-approximation) with respect to this metric in the Fréchet space coincides with the best approximation (ε-approximation) with respect to some seminorm.

The metrics on a Frechet space were constructed by S. Mazur [100], G. Birkhoff [18], S. Kakutani [76], 2 norm-like metric G. Albinus [3–5], D. Zarnadze [170,197] and others. These metrics are discussed in Section 2.5. Of the 7 different metrics, only the metric constructed by D. Zarnadze (see Section 2.5) has the following properties: 1. If we construct this metric on a Banach space, then its quasinorm coincides with the norm of this space, 2. the best approximation in the Frechet space with respect to this metric coincides with the best approximation with respect to some initial seminorm, 3. The Minkowski functional of metric balls is similar to the initial seminorms, 4. The metric preserves the geometry of the Frechet space. Moreover, the quasinorm of this metric is easy to calculate, since, unlike other metrics, it does not require calculating neither the exact upper bound of an infinite set, nor summing of infinite series.

In order to study the complex tasks of computational mathematics, it is necessary to have a close relationship between the balls of metrics on the Frechet space and the balls of semi-norms generating the topology. First of all, for unification study of approximation problems in Banach and Frechet spaces, it is necessary to have a metric on the Frechet space, which, if constructed on the Banach space, should coincide with its norm. It is also desirable that the balls of radius r metric as in a Banach space can be obtained by multiplying a unit ball of some seminorm from the sequence generating the topology on this radius. Such metric keeps the geometry of Frechet space. It was G. Albinus who proposed the construction of a metric maintenance of the geometry of the Frechet space $L^2_{loc}(\mathbb{R})$.

The quasi-norms $|x| = d(x, 0)$ of the mentioned metrics, in contrast to the norms $\|x\| = d(x, 0)$ of a Banach space, are no longer homogeneous and convex functional. At best, we can expect quasi-norms to be quasi-convex (their spheres will be absolutely convex). It is clear that on a non-normed Frechet space it is impossible to construct a metric with homogeneous quasi-norm. It is not known whether it is possible to construct a locally homogeneous metric (for any element of space there will be a numerical interval from which the condition of homogeneity of the quasi-norm for numbers taken from it will be satisfied). However, it should be noted that in the case of D. Zarnadze's metric, the quasi-norm is quasi-convex and for those elements of space for which $|x| \neq 2^{n-1}$, $n \in \mathbb{N}$, the local homogeneity condition of quasi-norm is fulfilled (see corollary of Theorem 2.5.7). More accurately, a metric with absolutely convex balls is constructed such that $\mathbb{R}^+$ represents the union of right half-open intervals, so that the balls corresponding to any two numbers from these intervals are similar to each other. Minkowski functional of balls of metric are similar of the original seminorms, i.e. are obtained from them by multiplying by a correspondingly chosen constant. Additionally, this metric is the simplest available metric and a monotone and quasi-convex quasinorm corresponds to it. These metric keeps the maintenance of the geometry of Frechet space.

"In the Banach case the basis of zero neighborhoods can be obtained as multiples of the unit ball. Therefore, the geometry of the unit ball is crucial in Banach space theory. In a Frechet space, however, the relation between different neighborhoods of zero is, in general, more important than in the local Banach spaces. This is the reason why the properties of the linking maps between the local Banach spaces are crucial in the theory of Frechet spaces" [21]. As noted above, contrary to this view, when studying best approximation problems the geometry of the unit ball of seminorms is also very crucial.

The problem of computing the ε-approximation of the solution operator with respect to the constructed metric is related to the problem of computing the ε-approximation with some seminorm. The abstract generalization of the ε-approximation was also discussed in the monographs ([158], pp. 25-26, [157], Section 3.2). The properties of the metric allowed us to define spline, central algorithms in the works [163, 167, 168, 170, 208] in the case when the solution operator, in contrast to the classical approach, acts between Fréchet spaces. This became possible by generalizing the linear problems studied in [158] to the case when not a single set of problem elements is considered, but their non-decreasing sequence [166]. We metrize the topology arising from this sequence with our metric. The resulting quasinorm is a generalized Minkowski functional, which acts as an analogue of the Minkowski functional of the set of the problem elements [167]. This approach allowed us to study problems that do not fit into the framework of Banach spaces and have not been considered before. In particular, classical equations with unbounded operators (QHO, Schrodinger, Tricomi, Lagrange, Legendre, Laplace-Beltram and other differential and integral equations) considered in Hilbert spaces are transferred to the Hilbert space of finite n-orbits and the Fréchet-Hilbert space of all orbits, where completely new phenomena related to infinite measurements are described (see Chapters 1, 4-6 of the monograph).

In classic monographs dedicated to Functional analysis and locally convex spaces, the Frechet spaces occupy an important place. To the theoretical issues of the theory of Frechet spaces in recent years have been devoted books of C. T. J. Dodson, G. Galanis, E. Vassiliou [44], D. Vogt [181], A. Kriegl [88], B. Dierolf [35]. The present monograph is an attempt to analyze the specificity of the above-mentioned computational processes in order to further their practical application.

Unlike Banach space, a (non-normable) Frechet space does not have absolutely convex and bounded zero neighborhoods. Furthermore, the strong dual to a Frechet space is normable if and only if the initial space is also normable. Strong dual to Frechet spaces are nonmetrizable (DF) spaces [65]. Therefore, the tasks of metrization, topological and geometric properties, best approximation theory and construction of algorithms of approximation theory in Frechet spaces became related to the study of properties of non-metrizable locally convex spaces. This connection became particularly emphasized in connection with the development of the theory of strict inductive limits of sequences of Banach, Hilbert, and Frechet spaces, and it is discussed in Chapter 2. The foundations of this theory were investigated by J. Dieudonne and L. Schwartz's works regarding the representation of topologies of nonmetrizable spaces of basic and generalized functions [43]. They posed problems, most of which were solved by A. Grothendieck, also by S. Dierolf [35] and D. Zarnadze. Strictly distinguished Frechet spaces were introduced, as the spaces whose strong dual space can be represented as a strict inductive limit of sequences of Banach spaces [193]. The term "strictly distinguished LCS" is analogous to A. Grothendieck's term "distinguished LCS". It should be especially noted that the question of investigation of Frechet spaces, the strong dual of which are strict inductive limits, was mentioned in the list of unsolved problems at the end of the work [43]. Subsequently, they say that all these problems were solved by A. Grothendieck [65], who considered (distinguished) Frechet spaces whose strong dual are the inductive limits of a sequence of Banach spaces. But in [65], the just named strictly distinguished Frechet spaces (quojections), which are intensively studied in Sections 2.2–2.4, were not considered.

Later, in studies carried out in the works of European mathematicians, instead of the term "strictly distinguished" different names: "strictly regular", "quojection" were used. In what follows instead of term "strictly regular", we will use the term "quojection" for Frechet spaces, which can be represented as a strict projective limit of a sequence of Banach spaces and whose strong dual space can be represented as a strict inductive limit of the sequence of Banach spaces. Because we have used the term (strictly regular) in our earlier works, we sometimes refer to them in parentheses for clarity. The results in this book will be formulated corresponding to this remark. We also note that based on the results in [14] and [199], strictly distinguished Fréchet spaces are a wider class than strictly regular Fréchet spaces, and it coincides with the class of pre-quojections (see Remark 2.3.1).

Quojections (Strictly regular Frechet spaces) appeared in our papers in connection with the problem posed by S. Mazur ([82], p. 366), to which the final answer in the case of Banach space was given by R. James [74]: Banach space is reflexive if and only if every continuous functional attains its supremum on the unit ball. In approximative form this means that Banach space is reflexive if and only if every hypersubspace is proximal. Let's give the necessary definitions:

Let (E, d) be a linear metric space, G be some (non-empty) subset in E and

$x \in E$. Moreover, under the linear metric space (E, d) we understand the real or a complex linear space whose dimension is greater than one and on which a translation-invariant metric d is defined so that linear operations are continuous. We need to find such an element $g_0 \in G$ for which the following equality holds $d(x, G) = \inf\{d(x, g);\ g \in G\} = d(x, g_0)$. The quantity $d(x, G)$ is called the best approximation of the element x through the elements of the set G, or the distance from the element x to the set G, as well the approximation error of element x by elements of set G. The element g_0 is called the element of the best approximation for the element x in the set G. A subset G is called proximal if for every $x \in E$ in G there is at least one element of the best approximation. A subset G has the uniqueness property if for each $x \in E$ in G there is at most one element of the best approximation. A subset G is called Chebyshev if it is proximal and has the uniqueness property. When solving best approximation problems, as well as any extremal problem, the following 5 questions arise: existence, uniqueness, establishment of characteristic properties of the best approximation element, calculation of the best approximation $d(x, G)$ and construction of an algorithm for finding the best approximation element. These questions were first posed by P. L. Chebyshev in the Banach space of continuous function on the segment with the uniform norm. The theory of best approximation in Banach spaces is presented in the monograph of I. Singer [148], as well as in the review articles of A. L. Garkavi [62] and V. M. Tikhomirov [155].

The monograph is in fact a continuation of monograph [158], in which the problem posed there ([158], p. 25) is studied with a generalized solution operator in Frechet spaces with the metric constructed by us. It contains original results that arose in the study of the problems of best approximation and ε-approximation in Frechet spaces. The spline and central algorithms taken out in the title of the monograph indicate the simplest and most accurate (strongly optimal) algorithms in Frechet spaces.

Approximation theories of continuous functions in Banach space $C[a, b]$ with Chebyshev norm and in Frechet space on open interval $C]a, b[$ with compact convergence topology, differ significantly. In particular, despite the fact that the set of polynomials in both of these spaces is everywhere dense, in the space $C]a, b[$ there is no analogue of Bernstein's theorem [5]. The generalization of the main results of Chebyshev's theory on space with the metric constructed by D. Zarnadze is given in the paper [206]. The best approximation of a continuous function on an open interval with respect to a metric coincides with an equal best approximation of this continuous function on some segment, except for some values. The best approximation element has the analogous property (see Examples 3.1.1–3.1.2 in Section 3.1).

One of the most important results of the theory of best approximations in Banach spaces is above mentioned James' theorem [73, 74], which has been established since the early 60s (James' theorem is a)⇔b), and the rest are its consequences): for a Banach space E the following statements are equivalent: a) The space E is reflexive. b) Every linear continuous functional $x' \in E'$ attains its norm on the unit ball of the space E. c) Every closed hypersubspace (i.e. a closed subspace of the codimension 1) of the space E is proximal, d) The space E has the proximality property, i.e. every closed subspace of E is proximal. e) Restriction of any linear continuous functional $x' \in E'$ on each closed subspace attains its norm on the unit ball of this subspace. f) In the space $E = F_1$ with the unit ball F there exists an interpolation spline for non-adaptive information of cardinality 1. g) In the space $E = F_1$ with the unit ball F there exists an interpolation spline for non-adaptive information of any cardinality.

Jame's theorem is presented in such a way (Theorem 1.3.1 is really a)⇔e)) that it becomes necessary and sufficient condition for the existence of an interpolation spline with non- adaptive information cardinalities 1. The existence of splines which depend on the existence of the best approximations in subspaces of finite codimension, makes it possible to construct spline algorithms, which in many cases turn out to be linear, optimal and even central ones (see Chapter one). Important for this are the concepts of the Chebyshev center and the Chebyshev radius of a the set which were introduced by A. L. Garkavi [56, 57].

In the case of Frechet spaces, the problem of generalization of James's theorem was considered in works [3–5, 53, 185, 193, 198]. It turns out that some propositions are no longer fair in the case of Frechet spaces. These issues are studied in the third chapter of the monograph for various metrics in connection with the existence of splines and spline algorithms. These issues were connected on solvability of the above mentioned problems of J. Dieudonne and L. Schwartz. In particular, in the case of proximality of all hypersubspaces of the Frechet space, that is, in the case of non-adaptive information of cardinality 1, a necessary and sufficient conditions for the existence of a spline are: the total reflexivity of the Frechet space and representation of its strong dual space as a strict inductive limit of a sequence of reflective Banach spaces. In such case Frechet space is a reflexive quojection. For the case of Frechet spaces is used to obtain necessary and sufficient conditions of existence spline for non-adaptive information of cardinality 1 (Theorems 3.2.1 and 3.2.2).

Remark 1. As we note above, if we construct our metric on a Banach space, then its quasinorm will coincide with the norm of a Banach space, and this property characterizes only it among the existing metrics. This allows us to extend existing results for Banach spaces to Frechet spaces using this metric, and to extend some existing norm-related notions for Banach spaces to Frechet spaces using metrics without changing the term. In our previous works we used the terms: generalized spline in Frechet space, generalized spline algorithm in Frechet space, generalized least squares method in Frechet space, generalized Ritz method in Frechet space, generalized central algorithm in Frechet space. Due to the known property of the metric we have constructed, we can mean that we are talking about a spline, spline algorithm, least squares method, Ritz method, about a central algorithm in Frechet spaces without the adjective “generalized”.

The case of the proximality of certain classes of subspaces of Frechet spaces E, as well as the conditions for the existence of splines in the case of non-adaptive information of finite cardinality, were described in terms of representing them and their strong dual spaces as strict projective and strict inductive limits of its strong dual space. As well, in the case of proximality of all subspaces of the Frechet space, that is, in the case of non-adaptive information of the arbitrary finite cardinality, a necessary and sufficient conditions are: the quojectiness of Frechet space and of its any quotient spaces, and representation of its strong dual space and of any of its subspaces as a strict inductive limit of a sequence of reflexive Banach spaces. This is the part of general fourth problem of J. Dieudonne and L. Schwartz, negatively solved by A.Grothendieck. We obtain the representation of such Frechet space as $B \times \omega$, where ω is the space of real or complex sequences R^N (C^N) (Corollary 2, Theorem 3.4.1).

Remark 2. Frechet–Hilbert spaces were introduced [194], as the spaces which can be represented as a strict projective limit of sequences of Hilbert spaces, and their strong dual space as a strict inductive limit of the same sequence of Hilbert spaces [201]. Representations of the topologies of basic and generalized function spaces are obtained, in which Frechet–Hilbert spaces and its strong dual spaces appear (see Section 2.6). This term was used in many works, in particular, [7,45,127,131, 171]. But in the work [131] the following is written: “The fundamental system of seminorms is not at all uniquely determined by the topology; any choice of such a collection of seminorms is called grading. A graded Frechet space is one equipped with a fixed choice of grading. Graded subspaces and graded quotient spaces are equipped with the induced gradings. A seminorm $\|\cdot\|$ on E is called hilbertian if there is a semi-inner product $(\cdot,\cdot)$ on E such that $\|x\| = (x,x)$, for all $x \in E$. A Frechet–Hilbert space is a Frechet space which admits a grading consisting of hilbertian semi-norms, a graded Frechet–Hilbert space is one equipped with such a grading”.

According this definition nuclear Frechet spaces and Frechet spaces with continuous hilbertian norms are graded Frechet-Hilbert spaces. In what follows, we will use the term “Frechet–Hilbert space” in this sense without the word “graded”.

Frechet spaces, which can be represented as a strict projective limit of a sequence of Hilbert spaces, in this book we call strict Frechet-Hilbert spaces. The results from paper [201] in this book will be formulated corresponding to this Remark 2.

Chapter 1 provides definitions of central, optimal, spline, and linear algorithms for a linear problem. These definitions are taken from [158]. Such algorithms in connection with various tasks were studied in the works of A. Sard, S. M. Nikolskij, J. F. Traub, H. Wozhnyakowski, G. W. Wasilkowski, N. S. Bakhvalov, S. A. Smolyak, K. Yu. Osipenko, V. V. Ivanov, A. G. Werschultz, C. A. Michelli, E. W. Packel, T. J. Rivlin, M. A. Kon, R. Tempo and others. Fundamental research on splines were carried out in the works of N. P. Korneychuk, V. M.Tikhomirov, S. B. Stechkin, A. A. Zensykbaev, D. Ugulava and others.

In Section 1.1, the definition of central, optimal algorithms for non-adaptive information of finite cardinality in Banach spaces of worst case error calculation are defined. Some results from the works of Garkavi [56] and [57] concerning the concept and existence of a Chebyshev center are given.

In Section 1.2, linear, optimal, spline and central algorithms for solution operators are defined and the results of Smoliak, Packel about existence of linear optimal algorithm are given. These notions were extended for equation $Au = f$, where $A : D(A) \subseteq B \to B$ is operator acting in Banach space. An example confirming that the classical method of integrating differentiable functions is a spline algorithm is given. It is proved that the Ritz algorithm in energetic spaces under some conditions is spline and central algorithm. It is also proved that the least squares method is a spline algorithm.

In Section 1.3, The conditions for the existence of splines in the case of information of cardinality 1 in Banach spaces are given by James's theorem and also by the Bishop–Phelps theorem. The theory of best approximation in subspaces of finite co-dimension, i.e. finite defect, in classical normed spaces is covered quite fully in the review article by A. L. Garkavi [62]. The existence of Chebyshev subspaces of a finite defect in the space $C(Q)$ (depending on the topological structure of Q) was studied by Phelps [122, 123], A. L. Garkavi [61].

In Section 1.4, n-orbit for linear, symmetric, positive definite operator with a discrete spectrum and dense image A at the point $x \in H$ is defined as a finite sequence $\mathrm{orb}_n(A, x) := (x, Ax, \dots, A^n x)$, $n \in \mathbb{N}_0$. The space of such elements is denoted by $D(A^n)$ and call the space of finite n-orbits. If A is a closed operator, then $D(A^n)$ turns into a Hilbert space, where H is an infinite-dimensional (real or complex) separable Hilbert space, using the inner product

$$\langle \mathrm{orb}_n(A, x), \mathrm{orb}_n(A, y)\rangle_n := (x, y) + (Ax, Ay) + \cdots + (A^n x, A^n y), \quad n \in \mathbb{N}_0.$$

The equation $Au = f$ for the space $D(A^n)$ takes the form $A_n(\mathrm{orb}_n(A, u)) = \mathrm{orb}_n(A, f)$, where $A_n : D(A^n) \to D(A^n)$ is the operator defined by equality

$A_n(\mathrm{orb}_n(A, u)) = \mathrm{orb}_n(A, Au)$. We call A_n as n-orbital operator, which corresponds to the operator A, and $A_n(\mathrm{orb}_n(A, u)) = \mathrm{orb}_n(A, f)$ is called the n-orbital equation in the space $D(A^n)$. The left inverse $S_n : H^{n+1} \to H^{n+1}$ to the operator A_n, i.e. the solution operator of the n-orbital equation is defined by the equality $S_n(\mathrm{orb}_n(A, Ax)) = \mathrm{orb}_n(A, x)$. In this section, a spline algorithm for the approximate solution of the n-orbital equation for non-adaptive information in the space $D(A^n)$ for the QHO, Schrodinger operator, for well-known differential operator, Laplace-Beltrami operator δ, are built.

In Section 1.4, for a separable Hilbert space H and a linear symmetric operator $A : D(A) \subset H \to H$ with discrete spectrum and dense image, an n-orbit at a point $x \in H$ is defined as a finite sequence $\mathrm{orb}_n(A, x) := (x, Ax, \ldots, A^n x)$, $n \in \mathbb{N}_0$. The space of such elements is denoted by $D(A^n)$ and is called the space of finite n-orbits. The equation $Au = f$ in the space $D(A^n)$ takes the form $A_n \,\mathrm{orb}_n(A, u) = \mathrm{orb}_n(A, f)$, where $A_n : D(A_n) \subset H^{n+1} \to H^{n+1}$ is the operator defined by the equality $A_n(\mathrm{orb}_n(A, u)) = \mathrm{orb}_n(A, Au)$. A_n is called the n-orbital operator corresponding to the operator A. In this section, a spline central algorithm for an approximate solution of the n-orbital equation in the space $D(A_n)$ is constructed. Applications are given for equations containing QHO, Schrödinger operators, Laplace-Beltrami δ, etc.

In Section 1.5, the equation $Ku = f$ is studied, where $K : H \to H$ is compact, injective, self-adjoint, positive operator. The n-orbit $\mathrm{orb}_n(K^{-1}, x) := (x, K^{-1}x, \ldots, K^{-n}x)$ $(n \in \mathbb{N})$ for the left inverse on the image $K(H)$ operator K^{-1} at the point x is considered. The operator K^{-1} is injective, selfadjoint, positive and is not continuous. The space $D(K^{-n})$ is defined analogously and is a subspace of H^{n+1}. This equation $Ku = f$ in the space $D(K^{-n})$ has the following form: $K_n \,\mathrm{orb}_n(K^{-1}, u) = \mathrm{orb}_n(K^{-1}, f)$, where $K_n : D(K_n) \subset H^{n+1} \to H^{n+1}$ is defined by the equality $K_n(u, K^{-1}u, \ldots, K^{-n}u) = (Ku, u, K^{-1}u, \ldots, K^{-n+1}u)$. This equation, generally speaking, is ill-posed, i.e., for this problem the solution operator $S_n = K_n^{-1}$ is unbounded. The operator K_n is symmetric and positive in the space $D(K^{-n})$. We call K_n the n-orbital operator corresponding to the operator K, and $K_n(\mathrm{orb}_n(K^{-1}, u)) = \mathrm{orb}_n(K^{-1}, f)$ the n-orbital equation. For an approximate solution of this n-orbital equation in the Hilbert space of finite n-orbits $D(K^{-n})$, a spline algorithm is constructed and an application is given for an approximate solution of the n-orbital equation corresponding to integral equations of the first kind.

In Section 1.6, the equation $Au = f$ is studied, where $A : H \to M$ is an operator acting from Hilbert space H to Hilbert space M and having a singular value decomposition (SVD) with respect to the orthogonal sequences ψ_k and φ_k. The operator $K = A^*A$ is self-adjoint, having positive eigenvalues, which correspond to the eigenvectors φ_k and we use the results of Section 1.5 for solution of the equation $A^*Au = A^*f$ in the space $D((A^*A)^{-n})$. We construct a spline algorithm for the approximate solution of equation in the space $D((A^*A)^{-n})$.

At the end of Chapter 1, in Section 1.7, the results of Section 1.6 are used for Radon transform that admits SVD. The Radon operator R of a function f defined on a p-dimensional Euclidean space $\mathbb{R}^p$ $(p \geq 2)$ maps integrals of f over all hyperplanes of $\mathbb{R}^p$. The main task of computed tomography is to reconstruct a function from its integrals over hypersubspaces.

Chapter 2 (Sections 2.1–2.2) provides basic definitions and facts from the theory of locally convex spaces and the theorem on the characterization of strict inductive limits of sequence of Banach spaces (strict (LB)-spaces) is proved.

In Section 2.3, the definitions and characteristics of those Frechet spaces are given the strong dual of which are strict (LB)-spaces (according to A. Grothendieck [65], the Frechet space is distinguished if its strong dual spaces are (LB)-spaces). In the work of D. N. Zarnadze [193] they were called strictly distinguished Frechet spaces. In the translations to English of earlier works of D. N. Zarnadze [194–197] such Frechet spaces are considered whose strong duals are strict inductive limits of canonical sequences of Banach spaces spanned on the polar of neighborhoods of zero.

They were called strictly regular Frechet spaces. It was proved that the class of strictly regular Frechet spaces coincides with a quojection, which was introduced by S. Bellenot and E. Dubinsky [15], as well with the class to relative complete B_0-spaces introduced by V. Slovikowski and V. Zavodowski [149] and with the class of Frechet spaces representable in the form of a strict projective limit sequences of Banach spaces. These spaces have been intensively studied since the early 80s and the term "quojection" became widespread for them. Although, the term "strictly regular Frechet space" introduced by us and the results we obtained about these spaces were repeatedly mentioned in works of S. Dierolf [36], S. Dierolf, V. B. Moscatelli [38], V. B. Moscatelli [111], J. Bonet [20], J. Bonet, J. Taskinen [27], J. Bonet, M. Maestre, V. B. Moscatelli, G. Metafune, D. Vogt [25], etc. Due to what was said in this monograph the term quojection will be used. In the work [14], E. Behrends, S. Dierolf and P. Harmand (see also V. B. Moscatelli [111]) showed that the strong dual of a Frechet space can be strict (LB)-space even in the case when Frechet space is not a quojection.

A. Grothendieck [65] proved that the strong dual of a strict (LB)-space is distinguished Frechet space. This result of A. Grothendieck is strengthened in Section 2.3, namely, it is proved that strong dual of a strict (LB)-space is quojection. It follows from this that the strong second dual to Frechet space, the strong dual of which is strict (LB)-space is a quojection. In the paper [103], the Frechet spaces whose strong second dual are quojections, were called pre-quojections. In [179], it was proved that the strong dual of a pre-quojection is a strict (LB)-space, i.e. a conversion of our above-mentioned result is obtained. That's why, unlike [199], we will use the term "pre-quojection" to denote the Frechet space whose strong dual is a strict (LB)-space in spite of the fact, that this space coincides the strict distinguished Frechet space. Various examples of quojection and specifications of the results obtained are given.

In Section 2.4, we consider strict Frechet–Hilbert spaces, i.e. quojections, representable as a strict projective limit sequences of Hilbert spaces. These spaces appeared in [194] when representing the topology of the space of generalized functions, as well as in the work of E. Kramar [84], when generalizing the concept of a self-adjoint operator in Frechet spaces (in [84] this space is called the H-Frechet space). In this section representations of Frechet-Hilbert spaces in the form of a strict projective limit of the sequence of its complemented subspaces, as well as representation of its strong dual space in the form of a strict inductive limit of the same sequence of its complemented subspaces are given.

It is also proved that Frechet–Hilbert spaces are characterized in the class of Frechet spaces by that every subspace of type $\operatorname{Ker} p$, where p is a continuous seminorm that has topological complement. Subspaces and quotient spaces of Frechet–Hilbert spaces and also their strongly conjugate space are studied. A sufficient condition is given in order that a closed subspace of the Frechet–Hilbert space had an orthogonal complement. It is proved that every finite-dimensional (one-dimensional) subspace of Frechet–Hilbert space has an orthogonal complement if and only if it is isomorphic to the Hilbert space.

In Section 2.5, a metric with the above properties is constructed on a metrizable LCS.

In Section 2.6, we study spaces that can be represented as strong inductive limit of sequence of Frechet spaces. We construct a metrizable locally convex topology on strong (LF)-space, which is weaker than initial and induces the original topology on each prelimit space. Various representations of strong topologies are given for (LF)-spaces and their strongly dual spaces. In particular, representations of the topology of basic and generalized functions spaces D and D' are given. In this case, the space D' is represented as the inductive limit of an uncountable family of Frechet-Hilbert spaces. Next, the complete countably normed Sobolev spaces of infinite order are defined. They are different from the Sobolev spaces of infinite order, which were introduced by Yu. A. Dubinsky [46]. Embedding Theorems of these spaces in the finite order Sobolev spaces are proved.

In Section 2.7, we study homomorphisms, i.e. linear, continuous and open operators between locally convex spaces E and F and their adjoint operators. It is investigated the stability problem of homomorphisms when changing the topology of the spaces E and F. This problem arose in Section 2.3 of Chapter 2 when researching the issue of existence of splines. Similar problems often arise in applications and they have been intensively studied since S.Banach. The most important results were obtained in the works of J. Dieudonne [42], J. Dieudonne and L. Schwartz [43], A. Grothendieck [66], G. Köthe [80, 83], F. Browder [29], V. S. Retah [141], V. P. Palamodov [119], K. Floret and V. B. Moscatelli [52], J. Bonet and J. A. Conejero [22], B. Dierolf [35]. In the papers [40, 204], a generalization of the well-known theorem of A. Grothendieck on homomorphism is given to the case of topologies that are not compatible with the dualities of the spaces E and F (Theorem 2.7.1).

Using this generalized theorem, we obtain necessary and sufficient conditions for a weak homomorphism to also be a homomorphism, if the spaces E and F are endowed with the strong topologies, Mackey topologies, topologies of strong precompact convergence and associated bornological topologies and others. Necessary and sufficient conditions are also obtained for the adjoint (second adjoint) mapping with respect to the weakly homomorphism to be again a homomorphism, when dual (bidual) spaces are endowed with different known topologies. There classes of pairs of locally convex spaces such that weak homomorphisms between them are homomorphisms in various topologies of these spaces are found. Also, classes of pairs of locally convex spaces, adjoint (second adjoint) to homomorphisms between which are again homomorphisms in various topologies of dual (second dual) spaces are found there. Applications of these results are given to known homomorphisms and open mappings. In particular, our results are applied to obtain the sufficient condition for openness and strong openness of a weakly open operator with respect to the research of F. Browder [29].

In Chapter 3, linear problem with a sequence of problem elements sets is considered. Due to the properties of the metric constructed in Section 2.5, in the case of a constant sequence of problem elements, this problem coincides with the linear problem from [158].

The study of the question whether these properties hold for Frechet spaces, is much more closely related to the study of topological and geometric properties of these spaces. To the generalization of this Theorem in the approximation form the works of K. Floret and M. Wriedt [53] and M. Wriedt [185] are devoted. Namely, in [53], it was shown that the famous James's Theorem is no longer valid for Frechet spaces. More precisely, an example of a reflexive, but not totally reflexive space of the Frechet-Montel type was built, in which for any norm-like metric there is non-proximal closed hypersubspace.

In [198], it was proved that in Frechet spaces from the proximality of all closed hypersubspaces, generally speaking, the proximality of all non-normed closed subspaces does not follow. In [3] (see also [5] and [4]), it was proved that the Frechet nuclear space of all number sequences $\omega = R^N$ (C^N) have the proximality property. The proximality of closed hypersubspaces in Frechet spaces with respect norm-like metrics has been studied in [53, 185, 193].

In Section 3.1, the definition of a interpolation spline and a spline algorithm is given. A generalization of James' theorem for the case of Frechet spaces is obtained: the exact class of Frechet spaces for which reflexivity is equivalent to the proximality of all closed hyperspaces is quojectivity (strictly regularity).

In Section 3.2, James' theorem for Frechet spaces is used to obtain necessary and sufficient conditions for the existence spline in the case of information of cardinality 1 (Theorems 3.2.1 and 3.2.2). In particular, it is proved that for the known norm-like and (2.5.4) metrics, this is equivalent to the proximality of all closed hypersubspaces, which in turn is equivalent to their strong proximality. From the necessary and sufficient conditions we have obtained, it follows that one of such topological conditions is reflexivity and quojectivity, i.e., that the space strong adjoint to a Frechet space be a reflexive strict (LB)-space. The examples of projective limits sequences of reflexive Banach spaces that are not quojections and therefore have non-proximal hypersubspaces are given.

In Section 3.3, there is also a generalization of the Bishop–Phelps theorem to the case of quejections and it is studied the problem of proximality and approximate compactness of finite-dimensional subspaces in Frechet spaces with respect to the metric (2.5.4).

In Section 3.4, necessary and sufficient conditions are also given in order that in Frechet space every non-normed closed subspace would be proximal with respect to the mentioned metrics. From this we see that the problem of finding of reflexive quejection having a non-proximal non-normable closed subspace is equivalent to the finding of a reflexive strong (LB)-space, some quotient space of which is not a strong (LB)-space, that is, parts of problem 3 posed by Dieudonné and L. Schwartz in [43], which was solved negatively by A. Grothendieck in [65]. It is proven also that the Frechet space has the proximality property if and only if it is isomorphic to the space $X \times \omega$, where X is a reflexive Banach space, and ω is the nuclear space of all real (complex) number sequences R^N (C^N). It follows from this that in the Frechet space $X \times \omega$ a spline exists for any non-adaptive information of any cardinality.

In Section 3.5, conditions for the existence of interpolation splines in the space of differentiable locally integrable functions in a square are established.

In the Chapter 4 of monograph generalization of the method of least squares and Ritz method for operator equation in the Frechet–Hilbert spaces is carried out.

The need for such generalizations is due to the fact that many differential and integral operators known in Hilbert spaces have been extended in spaces of generalized functions [63, 64, 194]. Let us take a closer look at this issue:

In the 50s of the 20th century, L. Schwartz carried out a rigorous mathematical substantiation of the theory of generalized functions, the foundations of which were laid in the works of P. Dirac and S. L. Sobolev. The results presented in this justification were based on the theory of strict (LF)-spaces previously created by J. Dieudonne and L. Schwartz. Namely, the space of compactly supported infinitely differentiable functions D was represented as a strict inductive limit of a sequence of Frechet spaces, and the space of distributions D' was defined as its dual space. It should be especially noted that A. Grothendieck [65] formulated the axioms of strong dual spaces to Frechet spaces ((DF)-spaces) and found representations of strong dual spaces to the space of strict (LF)-spaces (see Section 2.6).

One of the main achievements of the theory of generalized functions was that differential and integral operators were made free from the narrow framework of Banach and Hilbert spaces, in which they were not continuous, and in the space of generalized functions these operators became continuous. However, the above-mentioned spaces of basic and generalized functions turned out to be too complicated for applications because of the non-metrizability of these spaces. In this regard, we studied the topologies of the spaces D and D' (see Section 2.6) and obtained their various representations, in which, on the one hand, it is emphasized the importance of nuclear and countable Hilbert spaces and their strong dual (LH)-spaces (Theorems 2.6.1–2.6.5). On the other hand, when representing the topology of these spaces, important classes of strict Frechet–Hilbert spaces and their strong dual strict (LH)-spaces appeared (see Section 2.4).

These spaces in many cases play the same role as the spaces D and D'. Namely, for the continuity of the above operators in many practically important cases, it is sufficient to consider Frechet spaces and their strong dual spaces (Section 4.3), in which these operators are continuous. In particular, for an unbounded self-adjoint operator A with discrete spectrum in a Hilbert space H, we consider the Frechet space (of test functions) $D(A^\infty)$ and its strong dual space (of generalized functions) $D(A^\infty)'_\beta$ for which the restriction of the operator A to $D(A^\infty)$ and the extension of A to $D(A^\infty)'_\beta$ are continuous (Theorem 4.4.3). The appearance and importance of studying these spaces was due to the fact that many important spaces of functions (continuous, infinitely differentiable, analytic) on open domains, measures on locally compact spaces countable at infinity, regular generalized functions, as well as spaces with a countable number of restrictions near the boundary or at infinity (Schwartz spaces, Vladimirov algebras) in their natural topology belong to the indicated classes of Frechet spaces and their strong dual spaces. After intensive research by many famous mathematicians, various Frechet spaces of test and their strongly dual spaces of generalized functions emerged, in which unbounded operators already become continuous. Consequently, Frechet spaces and their strong dual spaces turned out to be a natural area for the action of differential and integral operators and the consideration of the corresponding operator equations.

In this regard, the need naturally arose to develop methods for approximate solutions of equations containing such operators in Frechet spaces. At a certain stage of development of theoretical research in the field of locally convex spaces, back in the middle of the 20th century, the idea was put forward of the development of computational mathematics in non-normed Frechet spaces and the use of the advantages that non-normed spaces have over Banach spaces [136]. On the other hand, the construction of approximate methods for solving operator equations in Frechet spaces was stimulated by practical problems in CT and medicine, which required solving operator equations in Frechet spaces. Namely, according to the L. Schwartz Theorem, the well-known Radon transform is a linear operator between Schwarz spaces, and the task of CT is inversion this operator [114]. The motivation for the development of computational mathematics in Frechet spaces was the Schwartz theorem on the actions of the Radon operator in Schwartz spaces. The problem of computational mathematics requires the construction of a computational algorithm for the inverse Radon operator.

It should be noted that the simplified Schwartz problem of computerized tomography that was decided by A. Cormack [32] in the 60s of the last century and found application in X-ray diagnostics (see Section 1.7 and Section 5.3). In 1979, A. Cormack and G. Hounsfield "for the development of computer tomography" were awarded the Nobel Prize.

In the monograph Z. Presdorff [134] many problems are formulated in terms of metric spaces like the Schwartz problem of CT, but meaningful results regarding approximate methods using the technique of Frechet spaces were not obtained. Z. Presdorff [134] Studied approximate methods for solving operator equations in countable Hilbert spaces by reducing them to a fixed Hilbert space. It consists in replacing an equation in Frechet spaces with "the same equation" in some fixed Hilbert space, but these equations are actually different from each other [134]. The equations we considered in Frechet space (in the projective limit of the sequence of Banach spaces) are replaced by projection equations in some (prelimit) Banach spaces from the specified sequence, the numbers of which depend on the dimension of the approximating subspaces. By increasing the dimension of subspaces, the space numbers, generally speaking, also increase and thus new computational processes arise that are not covered by the framework of Banach spaces and have not been considered until now.

Therefore, since the 60s of the last century, the search for methods for the approximate solution of the given equations in Frechet space has been intensively carried out. During this tame it became clear that the development of the theory of approximation methods in non-normed Frechet spaces should be based on the development of the best approximation theory. Indeed, the results obtained when solving problems of best approximations in various Banach function spaces underlie many computational algorithms of the projection method (spline, Ritz, least squares) for the approximate solution of well-posed and especially ill-posed problems of mathematical physics. In this regard, it should be noted that the best approximate solutions, quasi-solutions, and pseudo-solutions of ill-posed problems are elements of the best approximations in some subsets. Theorems 1.4.1–1.4.2 show that the theory of spline algorithms with non-adaptive information is nothing other than the theory of best approximation in subspaces of finite codimension and is the most important method for constructing central algorithms in Banach and Hilbert spaces [158].

In this monograph, the results of the theory of best approximation in Frechet spaces are used to create the foundations of the theory of a spline with non-adaptive information and a spline algorithm, which under some conditions are central (Theorems 1.4.1 and 4.4.5). They are also used to generalize known algorithms of the projection method (Ritz, least squares) and to approximate solution of various classes of operator equations in Frechet spaces (Theorem 4.3.4). Construction of the foundations of the theory of a spline algorithm with non-adaptive information in Frechet spaces, i.e. the study of the best approximations in subspaces of finite codimension in these spaces became possible after generalizing and strengthening the above-mentioned results of J. Dieudonne, L. Schwartz, and A. Grothendieck about quojection, whose strong dual is strict inductive limits of a sequence of Hilbert and Banach spaces. The generalization of spline algorithm and least squares method is achieved by developing the results of the theory of best approximations in the finite co-dimensional and finite-dimensional subspaces of the Frechet–Hilbert spaces.

This is precisely what the generalizations made in Chapter 3 concerning linear problems are devoted to, as well as the extensions of the least squares and Ritz methods implemented in Chapter 4.

In Section 4.1, method of least squares is extended to equation with an operator between Frechet–Hilbert spaces. Approximate solutions are obtained by minimizing the discrepancy relative to the metric, which in the Hilbert space case coincides with the metric generated by inner product. The uniqueness and convergence of a sequence approximate solutions to exact solution of equation is proved. A concrete realization of the obtained results is given in the case of continuously invertible and so-called tamely invertible operators mapping the Frechet spaces of power series of finite and infinite types, the Frechet spaces of rapidly decreasing sequences and the Frechet spaces of analytic functions defined on Stein's manifold into themselves.

In Section 4.2, a generalization of the Ritz method and the concept of symmetric and self-adjoint operators in Frechet–Hilbert space are given. The theory of symmetric operators in Hilbert spaces is most fully expounded in the monographs of M. Reed and B. Simon [138, 139], K. Iosida [71], F. Riesz and B. Sz.-Nagy [143], K. Moren [109], W. Rudin [146]. At the same time, the theories of bounded and unbounded operators on which the mathematical apparatus of QM is based differ significantly from each other. The foundations of this theory were first outlined in von Neumann's monograph [182]. On the other hand, the results of this theory go far beyond the needs of QM and they are widely used in various fields of mathematics and, in particular, mathematical physics and quantum mechanics. The development of the theory of generalized functions and, together with it, the theory of locally convex spaces made it possible to introduce various definitions of symmetric operators in some locally convex spaces in the works of N. N. Vakhania and V. I. Tarieladze [174] and E. Kramar [84].

Moreover, in [84] definitions of the adjoint to a continuous operator and a continuous self-adjoint operator were given in the case of strict Frechet–Hilbert spaces, and in [85] a theorem on the spectral representation of such a self-adjoint operator was proved. In this section, an extension of the concept of a self-adjoint operator without the requirement of continuity is given and an extension to the Frechet–Hilbert spaces of a number of basic theorems of the theory of self-adjoint operators is proved. In particular, the latter spaces contain important classes of countable Hilbert spaces, nuclear Frechet spaces, and strict Frechet–Hilbert spaces. Also known in the case of Hilbert spaces, Von Neumann's theorems on symmetrical and self-adjoint operators are generalized to the case of the mentioned Frechet–Hilbert spaces. A generalization is also given to the case of the well-known Hellinger–Toeplitz theorem [108] on the continuity of a symmetric operator defined over the entire space. In the case of positive definiteness, an analogue of the well-known Friedrichs theorem on an extension of such an operator is also proved.

In this section, the well-known Friedrichs theorem on the extension of a positive definite operator is generalized to the case of Fréchet–Hilbert spaces. The following result of von Neumann is well known: if T is a closed linear operator mapping an everywhere dense subset $D(T)$ of a Hilbert space H into itself and having a adjoint mapping T^*, then the orthogonal complements of the graphs $G(T)$ and $G(T^*)$ are the sets $V(G(T^*))$ and $V(G(T))$, respectively; the operators TT^* , T^*T, $I+TT^*$ and $I+T^*T$ are also self-adjoint, and the operators $I+TT^*$ and $I+T^*T$ also have continuous inverse operators. A generalization of this theorem is given to the case of Frechet–Hilbert spaces provided that $G(T)$ has the (H) property introduced by T. Precupanu [133].

Similar properties in the case of countable Hilbert spaces were studied by D. N. Zarnadze [190, 205], and in the case of locally convex spaces in the works of N. N. Vakhania and S. A. Chobanyan [31], G. Isak and V. Postolică [72] and in the review work of V. Postolică [132]. Examples of symmetric self-adjoint and positive definite operators are given. In fact, there are two types of examples: the first includes symmetric operators defined on the Hilbert space $L^2(\Omega)$ and extended to the space $L^2_{loc}(\Omega)$. The second includes examples of symmetric operators that are also symmetric on some countable Hilbert spaces. We note that continued operators in many cases become continuous even if they were not continuous before the continuation. It should be emphasized here that this situation differs from the known situations in which linear operators become continuous (for example: switching to a "graph topology", embedding a Hilbert space in the space of generalized functions, or considering a space with different weights).

At the end of Section 4.2, a sufficient condition is given for the extension of a symmetric operator defined in a Hilbert space to the strict Frechet–Hilbert space. An illustration is given of the application of this condition for the extension of the well-known position operator $Tx(t) = tx(t)$ from a dense subset $D(T)$ of the Hilbert space $L^2(\Omega)$ to the entire space $L^2_{loc}(\Omega)$. Similar structures were built in the works of A. V. Marchenko [98] and Krupa [89].

An extension of the Ritz method is given for equations with positive definite operators in the Frechet-Hilbert spaces. The energetic Frechet-Hilbert space E_A is defined and its representation is given. Next, the space E_A is considered by the metric (2.5.12) and the definition of the A-density of the sequence of basis functions in the mentioned Frechet-Hilbert spaces is given, similar to what was done in the monograph by G. I. Marchuk and V. I. Agoshkov [99]. A necessary and sufficient condition for the A-density of a sequence in the energetic space E_A of the operator A is proved. An approximate solution of the equations $Au = f$ is sought in the energetic space with respect to a certain energetic norm using the Ritz method. Obviously, in the case of Hilbert spaces, the above definition coincides with the classical one. We prove the existence of a sequence of approximate solutions constructed by the Ritz method, converging to a generalized solution of the equation $Au = f$. Some estimates are also given for the convergence of a sequence of approximate solutions to a generalized solution.

In Section 4.3, applications of the obtained results for the approximate solution of direct problems are given. More precisely, we consider the equation $Au = f$, where $A : D(A) \subset H \to H$ is an unbounded self-adjoint positive operator in the complex Hilbert space H. For this operator A the countable Hilbert space $D(A^\infty)$ is considered, which arises in connection with the distribution of the sequence of eigenvalues on the line (Theorems 4.4.2–4.4.3). These space were introduced by B. S. Mitjagin [108] (see also A. Pietsch [126]) and studied in detail in the works of H. Triebel [159, 160]. In these works $D(A^\infty)$ was the whole symbol, where A^∞, if taken separately, meant nothing. $D(A^\infty)$ is isomorphic to some subspace M of the space H^N and this isomorphism is obtained by the mapping

$$D(A^\infty) \ni x \leftrightarrow \operatorname{orb}(A, x) = (x, Ax, A^2x, \dots, A^nx, \dots) \in M \subset H^N.$$

The definition of the operator $A^\infty : D(A^\infty) \to D(A^\infty)$ is given by the equality $A^{\infty}\operatorname{orb}(A,x) = \operatorname{orb}(A,Ax)$ as restriction on $D(A^{\infty})$ of the operator $A^N : H^N \to H^N$ defined on H^N by equality $A^N\{x_k\} = \{Ax_k\}$, $\{x_k\} \in H^N$.

The notation $D(A^{\infty})$ takes on a new meaning and henceforth $D(A^{\infty})$ means the domain of definition of the operator A^{∞}. The continuity and self-adjointness of the operator A^{∞} in the Frechet–Hilbert space $D(A^{\infty})$, as well as the self-adjointness of the operator A^N in the strict Frechet–Hilbert space H^N are established. It is proved that if a sequence of eigenfunctions of the operator A that is orthogonal with respect to H is chosen as the basis functions, then the condition of Theorem 4.4.5 is satisfied for the non-adaptive information generated by the orbits of this sequence in the space $D(A^{\infty})$.

Therefore, the extended Ritz method in the case of the equation $A^{\infty}u = f$ in the Frechet–Hilbert space $D(A^{\infty})$ turns out to be spline algorithm and is linear and central. It is proved that the sequence of approximate solutions converges to the exact solution in the space $D(A^{\infty})$. This result is made concrete in the case of QHO, for which the space $D(A^{\infty})$ coincides with the Schwartz space $S(R)$. Next, applications of the obtained results are given for the approximate solution of equations containing differential operators of the Sturm-Liouville type, QHO, Legendre operator, Beltrami operator, etc.

Chapter 5 considers the ill-posed equation $Ku = f$ in the Hilbert space H for a compact self-adjoint operator K with positive eigenvalues. It is assumed that the conditions of existence and uniqueness are fulfilled, but the stability condition is not satisfied, i.e. the inverse operator K^{-1} is not continuous. In [156], for some ill-posed problems, a metric compact space E is considered, which the operator K maps onto itself isomorphically. Therefore, such equations in the space E have a unique stable solution. Similarly, we consider this ill-posed equation in a Frechet space E, to which the restriction of the operator K is an isomorphism of the space E onto itself. More precisely, the restriction of K to the Frechet space E, taking into account the topology, is a self-adjoint operator in E, which isomorphically maps the space E onto itself. To approximate solution the resulting equation in the metric Frechet space E, we use the Ritz method from Sections 4.2–4.3. A condition is given under which this method is a spline algorithm. In this case, the local error is equal to the local radius of information and is therefore minimal [158]. The results obtained in Section 5.2 were applied in the construction of a spline algorithm for the approximate solution of an equation with an operator admitting a SVD with respect to only orthogonal sequences. In Section 5.3, they were used to construct a spline algorithm for an approximate solution of the computer tomography problem. Recently, the construction of a spline for the Radon operator in Lizorkin space was the subject of work [115].

Chapter 6 describes the orbitalization process that we introduced. This then provides the foundations for a new mathematical model of quantum mechanics, which we call orbital quantum mechanics (of finite n-th order as well as infinite order). In this new theory, the operators expressing observable quantities receive a new interpretation.

By analogy with Dirac's "correspondence principle", QM is a special case of orbital quantum mechanics of order n, with $n = 0$. Using the technique of Hilbert spaces, we obtain laws and theorems of orbital quantum mechanics of order n, which are generalizations of known results. Orbital quantum mechanics of infinite order uses the technique of Frechet-Hilbert spaces and is a significant generalization in which the Schrödinger equation acquires a new meaning.

Section 6.1 presents the issues of quantization of classical physics and orbitization of quantum mechanics, i.e. the transition from QM to orbital quantum mechanics of n-order and to orbital quantum mechanics of infinite order, which are presented in the table. Orbital quantum mechanics studies orbits, orbital operators, orbital spaces and orbital equations for observable physical quantities of QHO, position and momentum. And also orbital operators corresponding to creation, annihilation and number operators. Each of the operators under consideration generates n-finite orbits and orbits in the state of quantum Hilbert space. They also generate n-finite orbital operators that act in the corresponding Hilbert spaces of finite n-orbits. These operators also generate orbital operators that act in the corresponding Fréchet-Hilbert spaces of all orbits.

In Section 6.2 we study n-orbits and infinite orbits of the QHO operators H, position X and momentum P in the states of the quantum Hilbert space, as well as n-orbital operators H_n, X_n and P_n corresponding to these operators. The norm of n-orbital space for position operator is expressed in terms of the mathematical expectation of the particle position. Some relations between the orbital operators H_n, X_n and P_n are also established. Generalized canonical commutation relations between X_n and P_n are proved, which coincide with the classical ones in the case $n = 0$. A representation of the expression for the orbital operator $X_nP_n + P_nX_n$ in terms of the Weyl quantization of a certain function is established. The orbits of the position and momentum operators in the states are studied, the Fréchet-Hilbert spaces of all orbits $D(H^\infty)$, $D(X^\infty)$ and $D(P^\infty)$ are introduced, as well as the orbital operators X^∞ and P^∞ in these spaces, and a generalized canonical commutation relation is proved. The connections between the orbital operators H^∞, X^∞ and P^∞ are also established. The Heisenberg uncertainty principle for orbital operators is proved and the issue of achieving equality in the Heisenberg inequality is considered.

In Section 6.3 finite orbits of the creation operator C and the annihilation operator A in the states of the quantum Hilbert space, as well as the n-orbital operator C_n and A_n corresponding to these operators in the Hilbert space of finite orbits are defined. We consider a generalization of the canonical commutation relations for the orbital operators C_n and A_n. Some relations between orbital operators Nn corresponding to numerical operator N and with the operators C_n and A_n are also established. The generalized canonical commutation relations between C_n and A_n are proved, which in the case $n = 0$ coincide with the classical ones. Orbits of creation and annihilation operators at states, the Frechet-Hilbert spaces of all orbits $D(C^\infty)$ and $D(A^\infty)$ the orbital operators C^∞ and A^∞ in these spaces are studied and generalized canonical commutation relation is proved. The analogous relationship between orbital operator N^∞, C^∞ and A^∞ is established.

In Section 6.4, for the approximation solution of the equation (6.4.6) a linear central spline algorithm in the space $D(A^\infty)$ is constructed. The convergence of the sequence of approximate solutions to the exact solution is proved. These results obtained for general operators are applied to the one-dimensional QHO operator in the Fréchet–Hilbert space of all orbits, which in this case coincides with the Schwartz space [170].

In Section 6.5, central spline algorithms for calculation of the inverse of multi dimensional Hamiltonian of QHO on Schwartz space are constructed.

Finally, we can conclude that by solving the problems posed by German mathematicians (G. Albinus, K. Floret, M. Wriedt), using and developing theories (strict inductive limits, spline and central algorithms, mathematical model of quantum mechanics) created by European and American scientists (J. Dieudonne, L. Schwartz, A. Grothendieck, P. Dirac, J. Traub, H. Wozniakowski and G. W. Vasilkovski), in this monograph the theory of linear, spline, central algorithms in the Hilbert and Frechet spaces of orbits is developed. These methods have been applied to equations containing orbital operators corresponding to Radon and Schrödinger operators in computerized tomography and orbital quantum mechanics.

The monograph uses the following system of references: Theorem 2.3.4 means the fourth theorem of Section 3 of Chapter 2, and the formula with the number (1.2.3) means the third formula of the second section of Chapter 1. When compiling the list of references, a general alphabetical order is used.

Obviously the attentive reader will find misprints and mistakes in the English translation and even errors. Thus we kindly ask to inform us about them – we would appreciate corrections.

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Since the 1980s, at the Institute of Computational Mathematics, under the leadership of academician N. Vakhaniya (1930–2014), there has been a seminar on the theory of spline and central algorithms, based on the books of American scientists A. Sard, J. F. Traub, H. Vozhniakowski and G. Vasilkowski. This seminar had a significant impact on the content of the monograph [163,165–168,170,203,205,208]. We would like to express our gratitude towards J. F. Traub (1932–2015), with whom we had discussions during his visit to Tbilisi on issues included in the monograph. They concerned the transfer of classical results of the theory of spline and central algorithms to Frechet spaces, which is implemented in this monograph. This connection is reflected in the article [169] published in the Journal of Complexity, founded by J. Traub in the USA.

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CHAPTER 1

Linear problems with an absolutely convex set of problem elements

1.1 Central and optimal algorithms for linear problems in Banach spaces

In this chapter we mainly use terminology and notation from [158]. Let F_1 be a linear space, G a normed space both over the same field of real or complex numbers R and C, respectively. Let $S : F_1 \to G$ be a linear operator, and F is a balanced and convex (absolutely convex) subset of the space F_1. According to [158], elements f from a problem elements set F are called problem elements, S is called the solution operator, and the elements $S(f)$ – are called solution elements. Let $U(f)$ – be the computed approximation of $S(f)$. The distance $\|S(f) - U(f)\|$ is called absolute error, where $\|\cdot\|$– is the norm in the space G.

Let $\varepsilon \geq 0$. We will say that $U(f)$ – is an ε-approximation $S(f)$, if $\|S(f) - U(f)\| \leq \varepsilon$. Our goal is to calculate a $U(f)$ such that which is the ε-approximation for all $f \in F$, i.e. $\sup\{\|S(f) - U(f)\|;\ f \in F\} \leq \varepsilon$. This formulation of the problem is called the worst case of error calculation.

How can we calculate such an approximation of $U(f)$? For this we need some knowledge about the problem elements. Let Λ denote a class of permissible information operators L. That is $L \in \Lambda$ iff $L(f)$ can be computed for each f from L. Let $y = I(f) = [L_1(f), \ldots, L_m(f)]$, where $L_1, \ldots, L_m \in \Lambda$ are linear functionals on F_1 (unlike [158], information is denoted by I, since N is accepted to denote the set of natural numbers). Such information in ([158], p. 27) is called non-adaptive. The number m is called the cardinality of the information $I(f) = [L_1(f), \ldots, L_m(f)]$. If $y = I(f)$ is known, then the approximation $U(f)$ calculated using this information gives us an element of the space G, which will be an approximation of the element $S(f)$. Therefore $U(f) = \varphi(I(f))$, where $\varphi : I(F_1) \to G$ is the mapping, which is called an algorithm.

For a given I, we want to construct $U(f)$ with a minimum error. In [158] such a problem, under the above suppositions regarding the space F_1, the set F and the operator S, is defined as a linear problem.

The approximation of $U(f)$ is calculated as $\varphi(I(f))$, where I is the information operator and φ is the algorithm which maps $I(F_1)$ to G . For given information I, the algorithms φ that are of interest are those that minimize the error

$$e(U) = \sup\{\|S(f) - U(f)\|,\ f \in F\} = \sup\{\|S(f) - \varphi(I(f))\|,\ f \in F\}.$$

Since U depends on φ and I, i.e. $U = (\varphi, I)$, it is convenient to denote $e(U)$ by $e(\varphi, I)$ and this quantity

$$e(\varphi, I) = \sup\{\|S(f) - \varphi(I(f))\|;\ f \in F\}$$

is called the global error for the algorithm φ that uses information I.

The local error for the algorithm φ is defined as follows: let $y = I(f)$, then the same element $\varphi(y)$ approximate all element of the set $SI^{-1}(y)$, where $SI^{-1}(y) = \{S(f) \in G; f \in I^{-1}(y)\}$ and $I^{-1}(y) = \{f \in F;\ I(f) = y\}$. Magnitude

$$e(\varphi, I, y) = \sup\{\|S(f') - \varphi(y)\|;\ f' \in I^{-1}(y)\}$$

is called the local error for φ. It's obvious that

$$e(\varphi, I) = \sup\{e(\varphi, I, y); y \in I(F)\}.$$

For a good approximation $U(f)$ we shall measure the set $SI^{-1}(y)$ by its radius. The radius of a subset M of a normed space G is number defined by

$$\operatorname{rad}(M) = \inf\{\sup\{\|c - g\|;\ c \in M\};\ g \in G\}\,.$$

It is the radius of smallest ball which contains the set M. A point $\mathfrak{m} \in G$ is called the Chebyshev center of a set M if

$$\operatorname{rad}(M) = \inf\{\sup\{\|c - g\|;\ c \in M\}; g \in G\} = \sup\{\|c - \mathfrak{m}\|;\ c \in M\}\,.$$

The radius of the set $SI^{-1}(y)$ is denoted by $r(I, y)$ and is called the local radius of information I at point y, i.e.

$$r(I, y) = \operatorname{rad}(SI^{-1}(y)) = \inf\{\sup\{\|a - g\|;\ a \in SI^{-1}(y)\};\ g \in G\}\,.$$

The global radius of information I is defined as the local radius for a worst y, i.e.

$$r(I) = \sup\{r(I, y);\ y \in I(F)\}.$$

Let the set $SI^{-1}(y)$ have a Chebyshev center $\mathfrak{m} = \mathfrak{m}(y)$ for all $y \in I(F)$. Then the algorithm φ defined by the equality $\varphi^c(y) = \mathfrak{m}(y)$ is called central.

In ([158], p. 50, Theorem 3.2.1) it is proved that the local and global radii of information I coincide with the exact lower bounds of the set of local and global errors relative to algorithms using information I, i.e.

$$r(I,y) = \inf\{e(\varphi, I, y);\ \varphi \in \Phi\},\quad y \in I(F), \quad \text{and} \quad r(I) = \inf\{e(\varphi, I);\ \varphi \in \Phi\},$$

where Φ is the set of all algorithms. The central algorithm φ^c, if it exists, minimizes global and local errors for all y.

An algorithm φ^* is called optimal error algorithm if

$$e(\varphi^*, I) = \inf\{e(\varphi, I);\ \varphi \in \Phi\}\,.$$

For the central algorithm we have $e(\varphi^c, I) = r(I)$, therefore the central algorithm is optimal. However, there are optimal algorithms ([158], p. 49) that are not central.

Let $y = I(f)$ be computed information. Using y we choose $\widetilde{f} \in F$ such that $\widetilde{f}$ interpolates y, i.e. $I(\widetilde{f}) = y$. Then the algorithm φ^i defined by equality

$$\varphi^i(y) = S(\widetilde{f})\,,$$

is called an interpolatory algorithm ([158], p. 51).

In ([158], p. 50), it is written: "In general, it is hard to find a center of a set, and so it is hard to obtain a central algorithm." In this monograph, we will construct the central algorithms for the important equations of functional analysis and quantum mechanics in the Hilbert and Fréchet–Hilbert spaces.

Let us present some results from the works of Garkavi [56] and [57] concerning the concept of a Chebyshev center.

Theorem 1.1.1. *There is a Banach space in which one can specify three points for which there is no Chebyshev center.*

Theorem 1.1.2. *If the space E is conjugate* (*to some normed space*), *then every bounded set $M \subset E$ has at least one Chebyshev center.*

Theorem 1.1.3. *Every normed space E can be isometrically and isomorphically embedded in some Banach space $\widetilde{E}$ so that for any bounded set $M \subset E$ in the space $\widetilde{E}$ there will be a Chebyshev center $\widetilde{y} \in \widetilde{E}$, and $\sup\limits_{x \in M} \|x - \widetilde{y}\|_{\widetilde{E}} = \frac{1}{2}\, d(M)$, where $d(M) = \sup\{\|m_1 - m_2\|;\ m_1, m_2 \in M\}$ is the diameter of the set M.*

For each separable space E, as $\widetilde{E}$ we can take the space of bounded measurable functions f on $[0, 1]$ (which can even be considered functions of at most 1st Baire class) with norm $\|f\| = \sup\{|f(t)|;\ \ t \in [0, 1]\}$. If E is nonseparable, then as $\widetilde{E}$ we can take the space of bounded functions measurable with respect to any measure, defined on the unit sphere of the conjugate space E', equipped with a weak topology (with a similar uniform norm).

The space E is said to be uniformly convex in every direction if for every element $z \in E$ and every $\varepsilon > 0$ there exists a number $\delta = \delta(z, \varepsilon) > 0$ such that if

$$x_1, x_2 \in E, \quad \|x_1\| = \|x_2\| = 1, \quad x_1 - x_2 = \lambda z \quad \text{and} \quad \|x_1 + x_2\| \geq 2 - \delta,$$

then $|\lambda| \leq \varepsilon$.

Theorem 1.1.4. *In order for each bounded set $M \subset E$ to have at most one Chebyshchev center, it is necessary and sufficient that the space E be uniformly convex in each direction.*

Theorem 1.1.5. *In order for each compact set $M \subset E$ to have at most one Chebyshchev center, it is necessary and sufficient that the space E be strictly convex.*

Studies related to finding Chebyshev center and building central algorithms for various tasks are carried out in the works [8].

1.2 Spline and linear algorithms. Examples

We can assume that the set F considered in Section 1.1 is generated by the restriction operator $T : F_1 \to X$, where $(X, \|\cdot\|)$ is the normed space and $F = \{f \in F_1, \|Tf\| \leq 1\}$. Indeed, the space X coincides with $\operatorname{Ker} \mu_F^{\perp}$, where μ_F is the Minkowski functional for the set F, given by the equality

$$\mu_F(f) = \begin{cases} 0, & \text{when } f = 0, \\ \inf\{\alpha;\ \alpha > 0,\ \alpha f \in F\}, & \\ \infty, & \text{when } f/\alpha \notin F,\ \forall \alpha \neq 0. \end{cases} \tag{1.2.1}$$

In this section, $\operatorname{Ker} \mu_F^{\perp}$ means the algebraic complement of the subspace $\operatorname{Ker} \mu_F$, $T : F_1 \to \operatorname{Ker} \mu_F^{\perp}$ is an algebraic projector, and the norm $\|\cdot\|$ on X is defined by the equality $\|Tf\| = \mu_F(f)$.

Let $I : F_1 \to R^m$ and $T : F_1 \to X$ be two linear operators, where F_1 is linear space and X – normed space, both above the field of complex numbers.

Let $y \in I(F_1)$. $\sigma = \sigma(y)$ is called a spline interpolatory y ([158], p. 95) if $I(\sigma) = y$ and

$$\|T\sigma\| = \min\{\|Tz\|;\ z \in F_1,\ I(z) = y\}.$$

The spline algorithm is defined by the equality

$$\varphi^s(y) = S(\sigma(y)),\ y \in I(F_1)\,, \tag{1.2.2}$$

where $\sigma(y)$ is a spline interpolatory y.

It is known ([158], p. 97) that according to this definition, the spline algorithm is interpolatory. It is homogeneous, but, generally speaking, it is not linear. A spline algorithm is uniquely defined if and only if when the set $S(P(T(f)))$ consists of one point for each $f \in F$, where

$$P(T(f)) = \{h \in \operatorname{Ker} I;\ \|Tf - Th\| = \inf\{\|Tf - z\|;\ z \in T(\operatorname{Ker} I)\}\}\,.$$

An interpolatory spline exists if and only if $P(T(f))$ is a non-empty set for all $f \in F$ for which $I(f) = y$.

The subset $G \subset E$ of a normed space $(E, \|\cdot\|)$ is called proximal with respect to the norm $\|\cdot\|$ if for all $f \in E$, in G there exists at least one element $g^* \in G$ of best approximation, i.e., such that

$$\inf\{\|f - g\|;\ g \in G\} = \|f - g^*\|.$$

The subset G is said to have the uniqueness property if for all $f \in E$ there exists at most one element of best approximation. The subset G is called a Chebyshev set if it is proximal and has a uniqueness property.

Theorem 1.2.1. *For the above mapping T, non-adaptive information I and for each $y \in I(F_1)$, an interpolatory spline exists if and only if the subspace* $\operatorname{Ker} I$ *is proximal in F_1 with respect to μ_F.*

Proof. Let for the mentioned T and for each $y \in I(F_1)$ there is an interpolation spline and $f \in F_1$. Let's assume that $I(f) = y_0$. For such y_0 there is a spline, i.e. an element $\sigma \in F_1$, such that $I(\sigma) = y_0$ and $f - \sigma \in P(Tf) = \{h^* \in \operatorname{Ker} I;\ \|Tf - Th^*\|\} = \inf\{\|Tf - Th\|;\ h \in \operatorname{Ker} I\}$. According to ([158], p. 57), we have that $\|Tf\| = \mu_F(f)$ for each $f \in F_1$. Therefore we also have that

$$f - \sigma \in P(Tf) = \{h^* \in \operatorname{Ker} I;\ \mu_F(f - h^*)\} = \inf\{\mu_F(f - h);\ h \in \operatorname{Ker} I\},$$

i.e. the subspace $\operatorname{Ker} I$ is proximal in F_1 with respect to μ_F.

Now let the subspace $\operatorname{Ker} I$ be proximal in F_1 with respect to μ_F and $y \in I(F_1)$. Let us show that for T and I there is interpolation spline. Consider the hyperplane $I(z) = y$. It has the form $f + \operatorname{Ker} I$, where $f \in F_1$ and $I(f) = y$. By condition, $\operatorname{Ker} I$ is proximal in F_1 with respect to μ_F and therefore there exists $h^* \in \operatorname{Ker} I$ such that $\mu_F(f - h^*) = \inf\{\mu_F(f - h); h \in \operatorname{Ker} I\}$. From this we obtain that $\sigma = f - h^*$ satisfies equalities $I(\sigma) = y$ and $\|T\sigma\| = \min\{\|Tz\|;\ z \in F_1,\ I(z) = y\}$. $\square$

Here we note that the definition of a spline and its basic properties in normed spaces are given in [158], P. M. Anselone, P.-I. Laurent [9], P.-I. Laurent [90], Ugulava [165].

It is natural to use algorithms that are convenient to use. Among these, linear algorithms with minimal errors stand out.

An algorithm φ^L is called linear using information $I(f) = [L_1(f), \dots, L_m(f)]$ if it has the form $\varphi^L(y) = \sum\limits_{i=1}^{m} L_i(f)q_i$, where $q_i \in G$ do not depend on f ([158], p. 75).

In the monograph ([158], p. 75), it is proved that linear algorithms with minimal error exist in the following three cases of linear problem:

(i) the range of a solution operator is R,

(ii) the range of a solution operator is suitable extended,

(iii) the range of a restriction operator is a Hilbert space and the image of the kernel of the information operator is closed.

Namely, the following propositions are proved:

Theorem 1.2.2 (Smolyak [150])**.** *Let S be a real linear functional and F be a balanced and convex set. Then there exist numbers q_i such that*

$$\Phi^L(I(f)) = \sum_{i=1}^{n} L_i(f)q_i$$

is an optimal error algorithm and

$$e(\Phi^L, I) = r(I) = \sup\{\|S(h)\|;\ h \in F \cap \ker I\}.$$

If $r(I) = +\infty$, any algorithm is optimal, and in this case in Theorem 1.2.2 arbitrary numbers can be taken as q_i. When $r = r(I) < +\infty$, then we can take $q_j = -\frac{c_j}{c_0}$, where $c_0, c_1, \dots, c_n$ are the coefficients of the reference plane to the set $Y = \{S(f), L_1(f), \dots, L_m(f) : f \in F\} \subset R^{m+1}$ at point $(r, 0, \dots, 0) \in R^{m+1}$.

Theorem 1.2.3 (Packel [118])**.** *Let a linear solution operator $S : F \to G$ be given. Then there exists a compact Hausdorff space D and the elements $q_i \in B(D)$, where $B(D)$ is the space of bounded scalar-valued functions defined on D with norm $\|g\| = \sup\{|g(x)|;\ x \in D\}$, such that*

(1) *G is isometrically isomorphic to a subspace $B(D)$;*

(2) $\Phi^L(I(f)) = \sum\limits_{i=1}^{m} L_i(f)q_i$ *is a linear optimal error algorithm for the solution operator* $\widehat{S} : F \to B(D)$*, where* $\widehat{S}(f) = \widehat{S(f)}$ *is the isometric image of* $S(f)$ *in* $B(D)$ *and*

$$\begin{aligned} e(\Phi^L, I) &= \sup\left\{\left\|\widehat{S(f)} - \Phi^L(I(f))\right\|;\ f \in F\right\} = r(I) \\ &= \sup\left\{\left\|\widehat{S(f)}\right\|;\ h \in F \cap \ker I\right\}. \end{aligned}$$

According to this theorem, a linear optimal error algorithm exists for an arbitrary linear problem if the range of a solution operator is extended to the space $B(D)$ with a suitably chosen topological space D.

The local error $e(\Phi, I, y) = \sup\{\|S(f) - \Phi(y)\| : f \in I^{-1}(y) \cap F\}$ of the algorithm Φ might be much greater than the local radius of information $r(I, y)$. This circumstance explains the interest in studying the magnitude

$$\operatorname{dev} \varphi(I) := \sup_{y \in I(F)} \frac{e(\varphi, I, y)}{r(I, y)},$$

which is called the deviation of the algorithm φ (in the case when $e(\varphi, I, y) = r(I, y) = 0$, for convenience, the value of the fraction on the right side is assumed to be equated to 1). The deviation of any algorithm is at least 1. It is desirable to have algorithms with low deviation. It is clear that these are central algorithms whose deviations are equal to 1, but their development and application is often too difficult, and often they might even not exist. On the other hand, linear algorithms are easy yo implement and it is natural to try to distinguish among them algorithms with small deviations. It turns out that spline algorithms are directly related to this issue.

Theorem 1.2.4 ([158], p. 97)**.** *Let* $SP(Tf)$ *be a one-point set in* X *for any* $f \in F$ *and let the radius of information* $r(I)$ *be finite. If the spline algorithm is linear, then the class of linear algorithms that use information* I *and have finite deviation consists of one element, namely the spline algorithm. If the spline algorithm is not linear, then the class of linear algorithms that use information* I *and have finite deviation is empty.*

From this theorem we can conclude that when constructing a linear algorithm with finite deviation, we must have a guarantee of the linearity of the spline algorithm. In particular, this occurs in the case when X is a Hilbert space and $T(\operatorname{Ker} I)$ is closed in X.

Theorem 1.2.5 ([158], p. 98)**.** *Let X be a Hilbert space, $T(\operatorname{Ker} I)$ be closed and let $r(I) < \infty$. Then for any linear solution operator S, the algorithm*

$$\varphi^s(I(f)) = \sum_{i=1}^{m} L_i(f) S\sigma_i \tag{1.2.3}$$

is a linear central algorithm, i.e. $\operatorname{dev}(\varphi^s, I) = 1$*, where* σ_i*,* $i = 1, 2, \ldots, m$*, be interpolation spline for* $y_i = [0, \ldots, \frac{1}{i}, \ldots, 0] \in R^m$*. Moreover, for any* $y \in I(F_1)$*,*

$$e(\varphi^s, I, y) = r(I, y) = \sqrt{1 - \|T\sigma(y)\|^2} \cdot r(I),$$

Where $r(I) = \sup\{\|S(h)\|/\|Th\|;\ h \in \operatorname{Ker} I\}$.

It is clear that under the conditions of theorem 1.2.5, the algorithm (1.2.3) coincides with the spline algorithm defined using the formula (1.2.2).

Note the result given in [158] with respect to the closedness of the set $T(\operatorname{Ker} I)$. It is proved in [158] that if F_1 is a Banach space, $\operatorname{Im} T$ is closed in X and $\operatorname{Ker} T$ has finite dimension, then the set $T(\operatorname{Ker} I)$ is closed in X.

Let us give an example confirming that the classical method of integration differentiable functions is a spline algorithm and, in some cases, a central one.

Example 1. Let us consider the problem of integration of non-periodic functions from the class $F_1 = W_p^r(0,1)$, where $r \geq 1$ and $p \in [1, \infty]$ ([158], p. 124). This class consists of functions $f : [0,1] \to R$, which have absolutely continuous derivatives $f^{(r-1)}$ of order $r-1$, and the derivatives $f^{(r)}$ of order r belong to the space $L_p[0,1]$. Let $X = L_p[0,1]$ with L_p-norm $\|\cdot\|_p$ and $Tf = f^{(r)}$. It is required to approximate $S(f) = \int_0^1 f(t)dt$ for f from $F = \{f \in F_1;\ \|Tf\|_p \leq 1\}$. Let Λ consists of linear functionals of the form $L(f) = f^{(j)}(t)$ for some $t \in [0,1]$ and $j \in [0, r-1]$, i.e. we have Birkhoff's information $I(f) = [f^{(j_1)}(t_1), \ldots, f^{(j_m)}(t_m)]$, $m \geq r$. For each $p \in [1, \infty]$the subspace $\operatorname{Ker} I$ is proximal into the space $L_p[0,1]$ with respect to the semi-norm $\mu_F(f) = \|Tf\|_p$ (the closedness of $T(\operatorname{Ker} I)$ in the space L_p follows from the embedding theorem). Therefore, by virtue of theorem 1.2.1, there is an interpolation spline σ whose rth derivative has a minimum $\|\cdot\|_p$ $-$norm. Therefore, to calculate $S(f)$, in the case of Birkhoff information, calculate

$$U(f) = \int_0^1 \sigma(t)dt\,.$$

In case $p = 2$, spline σ is a natural spline of order $2r-1$. Since σ is the center of symmetry of the set $I^{-1}(y) \cap F$ ([158], p. 97) and $S(\sigma)$ is the center of symmetry of the set $S(I^{-1}(y) \cap F)$, in the case of finite global radii, by ([158],

p. 50), σ is the Chebyshev center of the set $I^{-1}(y) \cap F$ in F_1 with respect to the seminorm $\mu_F = \|\cdot\|_2$ and $S(\sigma)$ is the Chebyshev center for the set $S(I^{-1}(y) \cap F)$ in $G = R$, i.e.

$$\inf_{s \in F_1} \sup_{f \in I^{-1}(y) \cap F} \|Tf - Ts\|_2 = \sup_{f \in I^{-1}(y) \cap F} \|Tf - T\sigma\|_2 = \mathrm{rad}(I^{-1}(y) \cap F),$$

and

$$\inf_{g \in R} \sup_{\widetilde{f} \in I^{-1}(y) \cap F} |S(\widetilde{f}) - g| = \sup_{\widetilde{f} \in I^{-1}(y) \cap F} |S(\widetilde{f}) - U(f)| = \mathrm{rad}(S(I^{-1}(y) \cap F).$$

It should be noted that the existence of spline and spline algorithm depends only on the proximality of $\mathrm{Ker}\, I$ in F_1 with respect to μ_F and does not depend on S unless F does not depend on S.

In what follows, we will call the operator S the solution operator for some equation $Au = f$ if $u = Sf$. If there is an inverse to the operator A, then $S = A^{-1}$. Further, the central (resp. linear, spline, optimal) algorithm that approximates the operator S will be called the central (resp. linear, spline, optimal) algorithm for the equations $Au = f$.

The following example of the Ritz algorithm in energetic spaces illustrates the above and is a spline algorithm. The following is a condition for its centrality.

Example 2. Let S be symmetric and positive definite operator in Hilbert space H with inner product $(\cdot,\cdot)$ and dense domain $D(S)$, i.e. in the above notation, $H = G$. Suppose H_S denotes the energetic space of the operator S and F_1 the linear space of elements belonging to H_S. Let $\{\varphi_i\}$ is a sequence of basis functions consisting of eigenfunctions of the operator S. Let us assume that the sequence of information $I(f) = [L_1(f), \ldots, L_m(f)]$ includes the functionals $L_i(f) = [f, \varphi_i]$, $i = 1, 2, \ldots, m$, where $[\cdot,\cdot]$ is the scalar product in H_S. $\mathrm{Ker}\, I = \{f :\in H : [f, \varphi_i] = 0,\ i = 1, 2, \ldots, m\}$ and its orthogonal complement is the subspace $\mathrm{Ker}\, I^{\perp} = span\{\varphi_1, \ldots, \varphi_m\}$. Let us assume that the set of problem elements F has the form $F = \{f \in H : [f] = [f, f]^{\frac{1}{2}} \leq 1\}$. As an operator T, consider the identical operator $T : F_1 \to H_S$ with F_1 on H_S. The subspace $T(\mathrm{Ker}\, I)$ is closed in H_S. Indeed, let $f \in \mathrm{Ker}\, I$ and $Tf_n \to f$ in H_S. This means that $[f, \varphi_k] = \lim\limits_{n\to\infty} [f_n, \varphi_k] = 0$, $k = 1, \ldots, m$, i.e. $f \in \mathrm{Ker}\, I$, $Tf \in T(\mathrm{Ker}\, I)$ and $T(\mathrm{Ker}\, I)$ is closed in H_S. By virtue of Theorem 1.2.1, to find the spline σ we must first find the best approximation h^* of the element f in the subspace $\mathrm{Ker}\, I$ of the space H_S with respect to the norm μ_F , and then put $\sigma = f - h^*$. Since the space (H_S, μ_F) is Hilbert, we can directly find the best approximation of the element f in the subspace $\mathrm{Ker}\, I^{\perp}$ with respect to μ_F. But it is well known [1] that the coefficients best approximation $u_m = \sum_{i=1}^{m} a_i \varphi_i$ of element f in the subspace $\operatorname{Ker} I^{\perp}$ satisfy the system of equations

$$\sum_{i=1}^{m} a_i[\varphi_k, \varphi_i] = [f, \varphi_k], \quad k = 1, 2, \ldots, m\,,$$

or

$$\sum_{i=1}^{m} a_i[\varphi_k, \varphi_i] = (Sf, \varphi_k), \quad k = 1, 2, \ldots, m\,.$$

In the case when we have the equation $Sf = g$ we get

$$\sum_{i=1}^{m} a_i[\varphi_k, \varphi_i] = (g, \varphi_k), \quad k = 1, 2, \ldots, m\,.$$

This system shows that the σ spline is nothing more than an approximate solution u_m, constructed using the Ritz method. It should also be noted that the spline algorithm

$$\Phi^s(y) = S(\sigma(y)),\ y \in I(F_1)$$

is linear.

According to theorem 1.2.5, if $r(I) < \infty$, then Φ^s is a central algorithm for local error as well

$$e(\Phi^s, I, y) = \sup\{\|S(f') - \varphi(I(f))\|;\ f' \in I^{-1}(y)\}$$

fair presentation

$$e(\Phi^s, I, y) = r(I, y) = (1 - \mu_{F}(\sigma(y)))^{1/2} r(I)\,,$$

where

$$\begin{aligned} r(I) &= \sup\left\{\frac{\|S(h)\|}{\mu_{F}(T(h))};\ h \in \operatorname{Ker} I\right\} \\ &= \sup\left\{\frac{\|S(h)\|}{(S(h), h)^{1/2}};\ h \in \operatorname{Ker} I\right\}\,. \end{aligned} \tag{1.2.4}$$

Therefore, it is natural to find out when the condition $r(I) < \infty$ is true.

Theorem 1.2.6. *Let S be a self-adjoint and positive definite operator in the Hilbert space H with dense domain of definition of $D(S)$. Then the following inequalities are valid:*

$$\|S\|^{1/2}_{\operatorname{Ker} I} \le r(I) \le (1/\gamma)\|S\|_{\operatorname{Ker} I}\,,$$

where

$$\|S\|_{\mathrm{Ker}\,I} = \sup\{\|S(h)\|;\ h \in \mathrm{Ker}\,I,\ \|h\| \le 1\}$$

and γ is a positive constant that participates in the definition of positive definiteness for the operator S.

Proof. From equality (1.2.4) due to the Cauchy–Buniakowski inequality

$$|(S(h), h)|^{1/2} \le \|S(h)\|^{1/2}\|h\|^{1/2}$$

we get the inequality

$$\begin{aligned} r(I) &= \sup\left\{\frac{\|S(h)\|}{(S(h), h))^{1/2}};\ h \in \mathrm{Ker}\,I\right\} \\ &\ge \sup\left\{\frac{\|S(h)\|}{\|(S(h)\|^{1/2}\|h\|^{1/2})};\ h \in \mathrm{Ker}\,I\right\} \\ &= \sup\left\{\frac{\|S(h)\|^{1/2}}{\|h\|^{1/2}};\ h \in \mathrm{Ker}\,I\right\} = \|S\|^{1/2}_{\mathrm{Ker}\,I}\,. \end{aligned}$$

On the other hand, from the positive definiteness of the operator S follows that there exists constant γ such that $(S(h), h) \ge \gamma\|h\|$. Therefore, we will have

$$\begin{aligned} r(I) &\le \sup\left\{\frac{\|S(h)\|}{\gamma\|h\|};\ h \in \mathrm{Ker}\,I\right\} = \\ &= (1/\gamma)\sup\{\|S(h)\|;\ h \in \mathrm{Ker}\,I \ \text{ i } \ \|h\| \le 1\} = (1/\gamma)\|S\|_{\mathrm{Ker}\,I}\,. \qquad \Box \end{aligned}$$

Corollary. *If the restriction of the operator S to $\mathrm{Ker}\,I$ is continuous, then $r(I) < \infty$.*

Example 3. The least squares method is a spline algorithm. Consider the equation

$$Au = f, \tag{1.2.5}$$

where A is a linear operator acting from a Hilbert space M into the same space N, such that there exists its continuous inverse operator A^{-1}. We denote this operator by S, i.e. $S(Ax) = A^{-1}(Ax)$ is the solution operator of the equation (1.2.5). Let F_1 be a linear space consisting of elements of the range $A(M)$ of the operator A. Let's assume that $A(M) = N$. In F_1, consider the set $F = \{Ax \in F_1;\ \|Ax\|_N \le 1\}$, where $\|Ax\|_N$ is defined as the Minkowski functional μ_F of the set F, i.e. $\mu_F(Ax) = \|Ax\|_N = (Ax, Ax)^{1/2}$, and $(\cdot,\cdot)$ is the inner product in N. $\mu_F(\cdot)$ is a norm on F_1 due to the existence and continuity of the operator A^{-1}. The space F_1 with such a norm will be denoted by $X = (F_1, \mu_F)$. Let $T : F_1 \to X$ be an identical operator, and $\{g_i\}$ some linearly independent system in M. Then the system $\{Ag_i\}$ is also linearly independent in N.

Let $I(f) = [L_1(f), \cdot, L_m(f)]$ be some non-adaptive information of cardinality m, where the linear and continuous on F_1 functionals $L_i(f)$, are defined as follows $L_i(f) = (Af, Ag_i),\ i = 1, \cdot, m.$

We have two operators: $T : F_1 \to X$ and $I : F_1 \to \mathbb{R}^m$. The kernel of the second $\operatorname{Ker} I := \{Ax;\ (Ax, Ag_i) = 0,\ i = 1, \ldots, m\}$ is a closed subspace of codimension m in $F_1 = A(M)$, and its orthogonal complement subspace is the set $\operatorname{Ker} I^{\perp} = \operatorname{span}\{Ag_1, \ldots, Ag_m\}$.

Let $y \in I(F_1)$. We are looking for an element $\sigma \in F_1$ such that $I(\sigma) = y$ and $\mu_F(T(\sigma)) = \min\{\mu_F(T(Ax));\ Ax \in F_1,\ I(A(x)) = y\}$.

The spline σ interpolatory y can be represented as $\sigma = x + h^*$, where x is some element from $I^{-1}(y)$, and h^* is an element of F_1 such that Th^* is an element of the best approximation for Tx in the subspace $T(\operatorname{Ker} I)$ of X. It is clear that h^* can be found as the element of the best approximation for Ax in $\operatorname{Ker} I^{\perp}$. From the theory of Hilbert spaces it is known that h^* has the form $h^* = \sum_{i=1}^m a_i Ag_i$, where the coefficients a_i, $i = 1, \ldots, m$ are determined from the system of equations

$$\sum_{i=1}^{m} a_i(Ag_i, Ag_k) = (Ax, Ag_k), \quad k = 1, \ldots, m.$$

If $x = u$, then we obtain that

$$\sum_{i=1}^{m} a_i(Ag_i, Ag_k) = (Au, Ag_k), \quad k = 1, \ldots, m. \tag{1.2.6}$$

On the other hand, according to the least squares method in space $X = (A(M), \mu_F)$, coefficients a'_i, $i = 1, \ldots, m$, of the approximate solution $u_m = \sum_{i=1}^m a'_i Ag_i$ of equation (1.2.5) are defined from the condition

$$\inf\{\|Au_m - f\|_N;\ u_m \in \operatorname{Ker} I^{\perp}\} = \inf\{\mu_F(TAu_m - Tf), u_m \in \operatorname{Ker} I^{\perp}\}.$$

It is known (see [1], p. 57) that these coefficients are determined from the system of equations

$$\sum_{i=1}^{m} a'_i(Ag_i, Ag_k) = (f, Ag_k), \quad k = 1, \ldots, m. \tag{1.2.7}$$

It follows from the equality $Au = f$ that the systems (1.2.6) and (1.2.7) coincide. This means that the least squares method is a spline algorithm for the solution operator $S = A^{-1}$, and has the form $\varphi^s(y) = S(\sigma(y))$.

1.3 Existence of splines for information of cardinality 1 in Banach spaces. Theorems of James and Bishop-Phelps

If $(F_1, \|\cdot\|)$ is a normed space with the unit ball F, then from Theorem 1.3.1 it follows that the theory of spline algorithms with non-adaptive information is the same as the theory of best approximation in subspaces finite codimension of the normed space F_1. In this case, $\mu_F(\cdot) = \|\cdot\|$ and the operator T will be the identity operator, i.e. the space X has the form $X = (F_1, \mu_F(\cdot))$.

In the case of subspace of codimension 1, i.e., for hypersubspaces, the most important results are the theorems of James [73], [74] and Bishop–Phelps [19].

Theorem (James [74])**.** *A Banach space E is reflexive if and only if every linear continuous functional on E attains its norm on the unit ball.*

In approximate form, this theorem states that a Banach space is reflexive if and only if every closed hypersubspace in it is proximal.

Theorem 1.3.1. *Let F_1 be a Banach space, F be unit ball of space F_1 and set of admissible functionals $\Lambda = F_1^*$, i.e., is algebraic dual space. For each $y \in R$ and each non-adaptive information I of cardinality $m = 1$ interpolation spline exists if and only if the space F_1 is reflexive. Moreover, in this case $\Lambda = F_1'$, i.e. set of admisible functionals cannot be wider than the space of linear continuous functionals and in the reflexive Banach space F_1 an interpolation spline exists for non-adaptive information of any cardinality $m \in \mathbb{N}$.*

Proof. Let $y \in R$. Consider the hyperplane $I(f) = y$. In our case it has the form $L(f) = y$, where $L \in F_1^*$. According to theorem 1.2.1 we have that each such hypersubspace is proximal with respect to μ_F. From the approximative form of James's theorem it follows that the space F_1 is reflexive. It follows that this hypersubspace is closed, and this is equivalent to the continuity of the functional L, i.e. $L \in F_1'$. From the reflexivity of the space F_1 we also obtain that interpolation spline exists for non-adaptive information $I(f) = [L_1(f), \ldots, L_m(f)]$, where $L_i \in F_1'$, because by theorem 1.2.1, this is equivalent proximality of the subspace $\operatorname{Ker} I$ in F_1 with respect to μ_F. $\square$

There is an example of an incomplete normed and therefore non-reflexive space, which was built by James, in which every closed hypersubspace is proximal.

Theorem (Bishop-Phelps [19])**.** *In a non-reflexive Banach space E, the set of proximal hypersubspaces constitutes an everywhere dense subset in the dual Banach space.*

We also note the work of S. I. Zukhovitsky [214], where the characterization of proximal hypersubspaces of the space $C(Q)$ are given.

The theory of best approximation in subspaces of finite codimension, i.e. finite defect, normed spaces is covered quite fully in the review article by A.L. Garkavi [62].

A fundamental complication of the problem of studying the approximation properties of subspaces of finite codimension different from unity is that for hyperspaces there are only two hyperplanes "parallel" to it, supporting the unit sphere, while for such subspaces of non-unit defect there are infinitely many planes.

An important means of studying infinite-dimensional subspaces were dual theorems that connected the problem of best approximation with the problem of extension of linear functionals. The first fairly general theorem of this kind was indicated by Phelps [123]: *in order that every linear functional defined on a subspace $L \subset X$ of a normed space has a unique extension without raising the norm to the entire space X , it is necessary and sufficient that the annihilator $($polar$)$ of the subspace $L^{\perp} \subset X^*$ be a Chebyshev subspace in the space X^*.*

The approximation side of this dual theorem concerns to the subspaces of the dual space X^*. For problems of approximation theory, dual theorems are of greater interest, the approximative content of which relates to subspaces of the original space X. A.L. Garkavi [58] obtained such theorems for the class of factor-reflexive subspaces, i.e. subspaces whose annihilators are reflexive subspaces in X^*. This class includes, in particular, subspaces of a finite defect. Garkavi's theorems characterize factor-reflexive subspaces that have the properties of existence and uniqueness in terms of extensions of functionals defined on the annihilators of these subspaces. The theorems of Phelps and Garkavi turned out to be useful in the study of subspaces of a finite defect, since in this case they reduced the infinite-dimensional approximation problem to a finite-dimensional extremal problem (such as the finite moment problem). Using dual theorems, Phelps [123] obtained a number of necessary or sufficient conditions for Chebyshev subspaces of finite dimension and finite defect. A.L. Garkavi [58] established a criterion for a subspace of a finite defect that was proximal. It was also shown there that Chebyshev subspaces of the defect $n < \infty$ can exist only in a Banach space whose sphere contains at least n linear independent extremal points.

Approximate properties of subspaces of a finite defect in the space $C(Q)$ were studied in the works of Phelps [123, 124] and A. L. Garkavi [58, 60]. The final results were obtained in the last two works, where the characteristic properties of subspaces having the properties of existence, uniqueness, and both together were established. Let us present one of the theorems from [61]. *In order for the subspace $L \subset C(Q)$ of defect n to be proximal, it is necessary and sufficient that the following conditions be satisfied:* a) *for any measure μ from the annihilator $L^{\perp} \subset C(Q)^*$*

there is a pair of closed sets that form a decomposition of its support $S(\mu)$ in the sense of Hahn; b) *for any measures μ_1, μ_2 from $L^{\perp}$ the set $S(\mu_1)\setminus S(\mu_2)$ is closed;* c) *the measure μ_1 is absolutely continuous with respect to μ_2 on $S(\mu_2)$.*

In the work [58] A.L. Garkavi obtained sufficient and some necessary criteria for Chebyshev subspaces of a finite defect in the space $L_1(Q, \mu)$.

The existence of Chebyshev subspaces of a finite defect in the space $C(Q)$ (depending on the topological structure of Q) was studied by Phelps [123, 124], A.L. Garkavi [61]. For the case of a metric compactum, the final result was obtained by A.L. Garkavi and is as follows: *for the existence of Chebyshev subspaces of defect $n > 1$ in the space $C(Q)$, it is necessary and sufficient that the compactum Q coincides with the closure of the set of its isolated points.* In this case the condition of "local disconnection" turned out to be necessary. It is proved that a necessary and sufficient condition for the existence of Chebyshev subspaces of defect n in the normed space $C_{L_1} \subset L(Q, \mu)$ of continuous integrable functions is the presence of at least n isolated points of the compact Q. In the works of Phelps [123] and A. L. Garkavi [60] obtained for classical spaces the characteristics of finite-dimensional subspaces that have the property of uniqueness of minimal extensions of all linear functionals. The latter work also established a general characteristic of such a subspace, which consists in the fact that at each non-zero point of the subspace the norm of the space X must be weakly differentiable with respect to some subspace complementary to the given one. The analytical form of this criterion is also given. These results, on the one hand, contain criteria for the uniqueness of a solution to the L-moment problem, considered by M. G. Krein [86], and on the other hand, by virtue of duality theorems, they give a characteristic of Chebyshev weakly closed subspaces in dual spaces.

The problem of best approximation with respect to Minkowski functionals and its dual problems were studied in detail in the works of A. D. Ioffe and V. M. Tikhomirov [70] and V. Ubhaya [164].

1.4 Linear central spline algorithm in space $D(A^n)$

Let H be a separable real or complex Hilbert space equipped with a norm $\|\cdot\|$, that is generated with inner product $(\cdot,\cdot)$, and $A : D(A) \subset H \to H$ be a linear, symmetric, positive definite operator with a discrete spectrum and dense image. The spectrum of A is called discrete if it consists of a countable set of eigenvalues with a single limit point at infinity.

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be a fixed nonnegative whole number and consider the elements of the space H, to which the operator $A^n = A(A^{n-1})$ can be applied, where A^0 is the identity operator. The space of such elements is denoted by $D(A^n)$, besides, $D(A^0) = H$. n-orbit of the operator A at the point $x \in H$ is a finite sequence $\operatorname{orb}_n(A, x) := (x, Ax, \ldots, A^n x)$, $n \in \mathbb{N}_0$. For the injective operator A, each orbit $\operatorname{orb}_n(A, x)$ is uniquely determined by the element $x \in H$, which we call the generating element of this orbit. The space $D(A^n)$ is identified with the space of n-orbits of the operator A. For simplicity of notation, sometimes the norm of the element $\operatorname{orb}_n(A, x) \in D(A^n)$, which is generated by the element $x \in H$, is denoted by $\|x\|_n$ instead of $\|\operatorname{orb}(A, x)\|_n$. If some operator B acts on $D(A^n)$, instead of $B(\operatorname{orb}_n(A, x))$ we will simply write Bx. We hope that such identification will not lead to misunderstandings. We can turn $D(A^n)$ into a pre-Hilbert space using the inner product

$$\begin{aligned}&\langle \operatorname{orb}_n(A, x), \operatorname{orb}_n(A, y)\rangle_n \\ &\qquad := (x, y) + (Ax, Ay) + \cdots + (A^n x, A^n y), \quad n \in \mathbb{N}_0, \quad (1.4.1)\end{aligned}$$

that generates the norm

$$\|x\|_n = \left(\|x\|^2 + \|Ax\|^2 + \cdots + \|A^n x\|^2\right)^{1/2}, \quad n \in \mathbb{N}_0. \quad (1.4.2)$$

If A is a closed operator, then $D(A^n)$ turns into a Hilbert space.

The equation $Au = f$ for the space $D(A^n)$ takes the form

$$A_n(\operatorname{orb}_n(A, u)) = \operatorname{orb}_n(A, f), \quad (1.4.3)$$

where the operator $A_n : D(A_n) = D(A)^{n+1} \subset H^{n+1} \to \operatorname{Im} A_n = (\operatorname{Im} A)^{n+1} \subset H^{n+1}$ is defined by the equality

$$A_0(u) = u,\ A_n(\operatorname{orb}_n(A, u)) = \operatorname{orb}_n(A, Au), \quad n \geq 1. \quad (1.4.4)$$

We call A_n as n-orbital operator, which corresponds to the operator A, and (1.4.3) is called the n-orbital equation in the space $D(A^n)$.

As noted above, if $n = 0$, then $D(A^0) = H$, $\langle x, y\rangle_1 = (x, y)$, and we have the equation $Au = f$ in the space H , i.e. classic case. For $n = 1$, $D(A)$ is considered according to the norm $\|x\|_2 = (\|x\|^2 + \|Ax\|^2)^{1/2}$ and if A is a closed operator, then $D(A)$ by the norm $\|x\|_2$ is a Hilbert space.

For the approximation solution of the equation (1.4.3), a linear spline central algorithm in the space $D(A^n)$ is constructed. The problem of convergence of the sequence of approximate solutions to the generalized solution is considered.

Let A be a symmetric positive definite operator in H with a dense image. We assume that A has a complete orthonormal sequence $\{h_k\}$ of eigenvectors, and the corresponding sequence of eigenvalues λ_k forms a discrete spectrum. The positive definiteness of A implies its injectivity, i.e. the existence on $\operatorname{Im} A$ of the left inverse operator S of A. To simplify the notation, we sometimes denote the left-hand side of the equation (1.4.3) by $A_n(u)$. This equation in coordinates has the form $A^{i+1}u = A^i f$ $(0 \le i \le n)$. The operator A_n defined by (1.4.4) is symmetric and positive definite in the space $D(A^n)$. Really, for arbitrary $u, v \in D(A^n)$ we have

$$\begin{aligned}\langle A_n u, v\rangle_n &= \langle (Au, A^2u, \dots, A^{n+1}u), (v, Av, \dots, A^n v)\rangle_n \\ &= (Au, v) + (A^2u, Av) + \dots + (A^{n+1}u, A^n v) \\ &= (u, Av) + (Au, A^2v) + \dots + (A^n u, A^{n+1}v) = \langle u, A_n v\rangle_n\end{aligned}$$

and

$$\begin{aligned}\langle A_n u, u\rangle_n &= (Au, u) + (A^2u, Au) + \dots + (A^{n+1}u, A^n u) \\ &\ge C(u, u) + C(Au, Au) + \dots + C(A^n u, A^n u) = C\langle u, u\rangle_n,\end{aligned}$$

where $C > 0$ exists according to the definition of positive definiteness of the operator A.

Let $\{h_k\}$, $k \in \mathbb{N}_0$, be an orthonormal basis on H. Next, we have

$$A_n(\mathrm{orb}_n(A, h_k)) = \mathrm{orb}_n(A, Ah_k) = \lambda_k \,\mathrm{orb}_n(A, h_k).$$

This means that $\mathrm{orb}_n(A, h_k)$ is the eigenvector of the operator A_n, corresponding to the eigennumber λ_k. The sequence $\{\mathrm{orb}_n(A, h_k)\}$ is orthogonal in $D(A^n)$, since

$$\begin{aligned}\langle \mathrm{orb}_n(A, h_k), \mathrm{orb}_n(A, h_i)\rangle_n &= (h_k, h_i) + (Ah_k, Ah_i) + \dots + (A^n h_k, A^n h_i) \\ &= (1 + \lambda_k\lambda_i + \dots + \lambda_k^n\lambda_i^n)(h_k, h_i) = 0 \;\text{ if }\; k \ne i.\end{aligned}$$

Moreover, the sequence $\{\mathrm{orb}_n(A, h_k)\}$ forms a complete system in the space $D(A^n)$.

The left inverse $S_n : H^{n+1} \to H^{n+1}$ to the operator A_n, i.e. the solution operator of the equation (1.4.3), is defined by the equality

$$S_n(\mathrm{orb}_n(A, Ax)) = \mathrm{orb}_n(A, x)$$

and is only positive compact operator on $\mathrm{Im}\, A_n$. S_n is the n-orbital operator corresponding to the operator S.

Our goal is to build a spline algorithm for the approximate solution of the n-orbital equation (1.4.3) in the space $D(A^n)$. For the construction of approximate solution $U(f)$, we apply some information about the problem element f. Let $y = I(f)$, $f \in D(A^n)$, be a nonadaptive information

$$I(f) = \big[(f, h_0)_n, (f, h_1)_n, \dots, (f, h_m)_n\big] \tag{1.4.5}$$

of the cardinality $m+1$.

Let us construct an interpolatory spline $y \in I(D(A^n))$ in the space $D(A^n)$ for the information (1.4.5). To do this, we consider the following spaces: the linear space F_1 consisting of elements of the space $D(A^n)$; $G = D(A^n)$ with the norm (1.4.3). Let T be an identical operator from F_1 on $X := (D(A^n), \|\cdot\|_n)$. The set of problem elements is $F = \{f \in F_1;\ \|T(f)\|_n \le 1\}$. The spline interpolatory $y = I(f)$ is defined as an element belonging to the space $D(A^n)$, which is generated by an element $\sigma_m \in H$ satisfying the conditions $I(\mathrm{orb}_n(A,\sigma_m)) = y$ and $\|T(\mathrm{orb}_n(A,\sigma_m))\|_n = \inf\{\|T(z)\|_n,\ z \in I^{-1}(y)\}$. According to the results of Theorem 1.2.1, $\mathrm{orb}_n(A,\sigma_m)$ is the best approximation element of $\mathrm{orb}_n(A,f) \in D(A^n)$ in the orthogonal complement subspace $(\mathrm{Ker}\, I)^\perp \subset D(A^n)$ with respect to the Hilbertian norm $\|\cdot\|_n$, and has the form

$$\begin{aligned}\mathrm{orb}_n(A,\sigma_m) &= \sum_{k=0}^{m} \frac{(f,h_k)_n}{(h_k,h_k)_n}\,\mathrm{orb}_n(A,h_k)\\ &= \sum_{k=0}^{m} \frac{(1+\lambda_k^2+\cdots+\lambda_k^{2n})(f,h_k)}{(1+\lambda_k^2+\cdots+\lambda_k^{2n})(h_k,h_k)}\,\mathrm{orb}_n(A,h_k)\\ &= \sum_{k=0}^{m} (f,h_k)\,\mathrm{orb}_n(A,h_k). \qquad (1.4.6)\end{aligned}$$

The coefficient in (1.4.6) does not depend on n. This means that the element $\mathrm{orb}_n(A,\sigma_m) \in D(A^n)$ is a spline that simultaneously corresponds to the information $y = [(f,h_0)_n, \ldots, (f,h_m)_n]$ as well as to the information $y_1 = [(f,h_0), \ldots, (f,h_m)]$. The spline algorithm is defined by the equality $\varphi^s(y) = S_n\,\mathrm{orb}_n(A,\sigma_m)(y)$, where $\mathrm{orb}_n(A,\sigma_m)(y)$ is a spline interpolatory y.

Taking into account the equality $S_n\,\mathrm{orb}_n(A,h_k) = \lambda_k^{-1}\,\mathrm{orb}_n(A,h_k)$, we obtain

$$\begin{aligned}\mathrm{orb}_n(A,u_m) = S_n\,\mathrm{orb}_n(A,\sigma_m) &= S_n \sum_{k=0}^{m} (f,h_k)\,\mathrm{orb}_n(A,h_k)\\ &= \sum_{k=0}^{m} \lambda_k^{-1}(f,h_k)\,\mathrm{orb}_n(A,h_k), \qquad (1.4.7)\end{aligned}$$

where S_n is the solution operator of the equation (1.4.3). This means that $U(f) = \mathrm{orb}_n(A,u_m) = S_n\,\mathrm{orb}_n(A,\sigma_m)$ is a spline algorithm for the information (1.4.5), where $L_i(f) = (f,h_i)_n$. It is well known [**?**] that the sequence of the approximative solutions $\mathrm{orb}_n(A,u_m)$ converges to generalized solution $\mathrm{orb}_n(A,u_0)$ of the equation (1.4.3) in the space $D(A^n)$.

Let us prove now that the algorithm (1.4.7) is central. We carry out the proof by analogy of ([158], p. 97). The spline $\mathrm{orb}_n(A, \sigma_m)$ interpolatory $y = I(f)$ is given by (1.4.6). We prove that it is the center of the set $I^{-1}(y) \cap F$. Let $f \in I^{-1}(y) \cap F$ and $f \neq \sigma_m$. We have $2\sigma_m(y) - f \in I^{-1}(y)$ and $\|T(2\sigma_m - f)\|_n = \|T(\sigma_m - h)\|_n$. Th is the best approximation element for Tf in the subspace $T(\mathrm{Ker}\, I)$ and, therefore, $(T\sigma_m, Th)_n = 0$. Now, from the latter equality we obtain

$$\|T(2\sigma_m - f)\|_n = \sqrt{\|T\sigma_m\|_n^2 + \|Th\|_n^2} = \|Tf\|_n \leq 1.$$

Thus, f and $2\sigma_m - f$ belong to the set $I^{-1}(y) \cap F$ and $\sigma_m = (f + (2\sigma_m - f))/2$. This means that σ_m is the Chebyshev center of the set $I^{-1}(y) \cap F$, i.e.

$$\begin{aligned}&\inf\{\sup\{\|c - a\|_n,\ c \in I^{-1}(y) \cap F\};\ a \in D(A^n)\}\\ &\qquad\qquad = \sup\{\|c - \sigma_m\|_n,\ c \in I^{-1}(y) \cap F\}.\end{aligned} \tag{1.4.8}$$

Further, we have that $S_n(\sigma_m) = \mathrm{orb}_n(A, u_m)$ is the Chebyshev center for $S_n(I^{-1}(y) \cap F)$ and, according to Theorem 1.2.5, the following equalities are valid:

$$\begin{aligned}e(\varphi^s, I, y) &= r(I, y) = \inf\{\sup\{\|a - g\|_n,\ a \in S_n(I^{-1}(y) \cap F)\};\ g \in D(A^n)\}\\ &= \sup\{\|a - u_m\|_n,\ a \in S_n(I^{-1}(y) \cap F)\}\\ &= (1 - \|\sigma_m\|_n)^{1/2} r_n(I),\end{aligned} \tag{1.4.9}$$

where

$$r_n(I) = \sup\{\|S_n(h)\|_n / \|T(h)\|_n,\ h \in \mathrm{Ker}\, I\}.$$

We note that $r_n(I) < \infty$ if and only if the restriction of the operator S_n on $\mathrm{Ker}\, I$ is continuous. It is easy to see that this is valid for compact positive operator S_n. Thus, the algorithm (1.4.7) is central for the information (1.4.5). We have proved that in the above notation the following statement is valid.

Theorem 1.4.1. *Let H be a separable Hilbert space, A be a symmetric, positive definite operator with dense image in H, and the operator A is closed. We will require that the set $T(\mathrm{Ker}\, I)$ is closed and the radius of information I is finite. Then the algorithm* (1.4.7) *is linear central spline algorithm for the approximate solution of the n-orbital equation* (1.4.3) *in the space of n-orbits $D(A^n)$. The sequence of the approximative solutions $\{\mathrm{orb}_n(A, u_m)\}$ converges to the generalized solution $\mathrm{orb}_n(A, u_0)$ of the equation* (1.4.3) *in the space $D(A^n)$.*

The received results we apply for the quantum harmonic oscillator operator $Au(t) = -u''(t) + t^2 u(x)$, $t \in \mathbb{R}$, in the Hilbert space of finite orbits $D(A^n)$.

First, we consider the equation (1.4.3) in the space $D(A^n)$, when $H = L^2(\mathbb{R})$ and A is the quantum harmonic oscillator whose domain includes functions $u(t)$ satisfying the condition $t^i u^{(j)}(t) \in L^2(\mathbb{R})$, $i, j \geq 0$, $i + j \leq 2n$, $j \leq 2n-1$, and, moreover, are zero at infinity ($u^{(j)}$ denotes the $j-$th derivative of u). It is easily verified that A is symmetric.

The eigenfunctions of the operator A are the Hermite functions (wave functions of a harmonic oscillator) ([178], p. 115):

$$h_j(t) = (-1)^j (j!)^{-1/2} 2^{-j/2} \pi^{-1/4} e^{t^2/2} \frac{d^j e^{-t^2}}{dt^j}, \tag{1.4.10}$$

where $j \in \mathbb{N}_0$. The spectrum of A is discrete and the eigenvalues of the operator A are $\lambda_j = 2j + 1$, $j \in \mathbb{N}_0$. The sequence $\{h_j\}$ forms an orthonormal basis of the space $L^2(\mathbb{R})$. The harmonic oscillator A is symmetric and positive definite.

Theorem 1.4.2. *Let* (1.4.3) *be the equation in the space* $D(A^n)$*, where* A_n *is the orbital operator corresponding to the quantum harmonic oscillator* A *in the space* $L^2(\mathbb{R})$ *and* I *be the information* (1.4.5). *Then the algorithm* (1.4.7) *is linear central spline algorithm for the approximate solution of the* n*-orbital equation* (1.4.3) *in the space of* n*-orbits* $D(A^n)$. *The sequence of the approximate solutions* $\{\mathrm{orb}_n(A, u_m)\}$ *converges to the generalized solution* $\mathrm{orb}_n(A, u_0)$ *of the equation* (1.4.3) *in the space* $D(A^n)$.

Proof. It remains to prove that the space $T(\mathrm{Ker}\, I)$ is closed, but this follows from the fact that the information functionals are generated from the elements of basis and are continuous for every $n \in \mathbb{N}$. □

Remark 1.4.1. The method described above can be applied to the equations containing the Schrödinger operator in spaces of finite orbits. The Schrödinger equation is the basic equation of quantum mechanics [128] and is usually written as

$$\left\{ -\frac{\hbar^2}{2m}\frac{d^2}{dt^2} + V(t) \right\} \psi(t) = E\psi(t),$$

where $\hbar$ is Planck's constant, m is the mass of the particle, $V(t)$ is the potential energy of the particle in the force field at position t, E is total energy, $\psi(t)$ is the wave function, d^2/dt^2 is the rate of change of the slope (the curvature) of the wave function at the position t.

The Schrödinger equation can be written even more compactly by defining of the so-called hamiltonian $\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dt^2} + V(t)$. With this definition, the equation becomes

$$\mathcal{H}\psi = E\psi. \tag{1.4.11}$$

In discrete spectrum problem, the eigenfunctions ψ_j and the respective energies E_j have to be determined by solving the eigenvalue problem

$$\frac{d^2\psi_j}{dt^2} + \frac{2m}{\hbar^2}[E_j - V(t)]\psi_j = 0, \quad \int\limits_{-\infty}^{\infty} |\psi_j|^2 dt = 1,$$

$\psi_j \to 0$ at $t \to \pm\infty$. In the case $V(t) = m\omega^2 t^2/2$ (ω-frequency of oscillator),

$$E_j = \hbar\omega(j + 1/2), \quad j \in \mathbb{N}_0.$$

Normalized eigenfunctions

$$\psi_j(t) = \frac{\pi^{-1/4}}{(2^j \cdot j! t_0)^{1/2}} \exp(-\xi^2/2) H_j(\xi), \quad \xi = t/t_0, \quad t_0 = \sqrt{\frac{\hbar}{m\omega}},$$

where H_j are Hermite polynomials. The functions $\psi_j(t)$ form an orthonormal basis in $L^2(\mathbb{R})$.

The obtained results can easily be extended to the equation (1.4.11), where the right-hand side is replaced by an arbitrary element f of the orbital space. More precisely, we can construct a linear spline central algorithm for the orbital equation $\mathcal{H}_n \operatorname{orb}_n(\mathcal{H}, \psi) = \operatorname{orb}_n(\mathcal{H}, f)$ with the orbital operator $\mathcal{H}_n$ of hamiltonian in the Hilbert space of finite orbits $D(\mathcal{H}^n)$.

The results obtained can be applied to essentially self-adjoint and positive definite operators $A_{m,k}$ $(2k \le m)$ ([160], 7.4.1) and Tricomi operators $B_{n,k}$ ([160], 7.6.3).

We give examples of selfadjoint and positive definite operators in Hilbert spaces that satisfy the conditions of Theorem 1.4.1. These examples are mainly taken from [160].

2. Consider the differential operator ([106], Chapter 5, Section 9)

$$Bu = -\frac{1}{t}\Big[\frac{d}{dt}\Big(t\frac{du}{dt}\Big) - \frac{\nu^2}{t}u\Big], \quad \nu = const > 1/2, \quad 0 < t < 1,$$

in the space $H = L_2(t; 0, 1)$ of functions quadratically summable on $(0, 1)$ with weight t. The domain of definition of $D(B)$ consists of functions u for which: $u(t)$ and $u'(t)$ are absolutely integrable on the interval $[\varepsilon, 1]$ $(0 < \varepsilon < 1)$; $\sqrt{t}u'(t)$ is continuous on $[0, 1]$ and vanishes at $t = 0$; $Bu \in H$ and $u(1) = 0$. In ([106], Chapter 5, Section 9) it was proven that $D(B)$ is dense in H, the operator B is symmetric and positive definite in H and has a discrete spectrum. The eigenvalues of operator B are

$$\lambda_k = j^2_{\nu,k}, \quad k \in \mathbb{N}, \tag{1.4.12}$$

where $j_{\nu,k}$ is the kth positive root of the Bessel function $J_\nu(t)$; the corresponding orthonormal eigenfunctions have the form

$$\varphi_k(t) = \frac{\sqrt{2}}{J_{\nu+1}(j_{\nu,k})} J_\nu(j_{\nu,k}t), \quad k \in \mathbb{N}. \tag{1.4.13}$$

The equation $Bu = f$ in the space $D(B^n)$ have the following form

$$B_n \operatorname{orb}_n(B, u) = \operatorname{orb}_n(B, f), \tag{1.4.14}$$

approximate solutions of which are

$$\operatorname{orb}_n(B, u_m(t)) = \sum_{k=1}^{m} \lambda_k^{-1} \int_0^1 s f(s)\varphi_k(s) ds \ \operatorname{orb}_n(B, \varphi_k(t)),$$

where

$$\begin{aligned} \operatorname{orb}_n(B, \varphi_k(t)) &= \big(\varphi_k(t), B\varphi_k(t), \dots, B^n\varphi_k(t)\big) \\ &= \big(\varphi_k(t), \lambda_k\varphi_k(t), \dots, \lambda_k^n\varphi_k(t)\big), \end{aligned}$$

λ_k and φ_k are defined according to (1.4.12) and (1.4.13). The sequence $\{\operatorname{orb}_n(B, u_m)\}$ converges in the space $D(B^n)$ to a solution of the equation (1.4.14) if $\operatorname{orb}_n(B, f) \in D(B^n)$. For the sequence of approximate solutions $\{\operatorname{orb}_n(B, u_m)\}$ Theorem 1.4.1 is valid in the space $D(B^n)$ with norm (1.4.2), in which A is replaced by B.

3. Laplace-Beltrami operator δ. Let S be the unit sphere in the l-dimensional Euclidean space $\mathbb{R}^l$, $\vartheta_1, \vartheta_2, \dots, \vartheta_{l-1}$ spherical coordinates of the point $\theta \in S$ and $\Sigma = \{t : \ \rho_1 \le |t| \le \rho_2, \ t \in R^l\}$, where ρ_1 and ρ_2 are arbitrarily fixed positive numbers such that $\rho_1 < 1 < \rho_2$. Consider a function f that is defined on S and let $f^*(t) = f(t/|t|)$ be the extension of f to $\sum$. We say that a function f belongs to the class $C^{(2)}(\sum)$ if all second-order derivatives of f^* are continuous on $\sum$. The operator δ is defined on $C^{(2)}(S)$ as

$$\delta = -\sum_{j=1}^{l-1} \frac{1}{q_j \sin^{l-j-1}\vartheta_j} \frac{\partial}{\partial\vartheta j}\Big(\sin^{l-j-1}\vartheta_j \frac{\partial}{\partial\vartheta j}\Big),$$

where $q_1 = 1, \ q_j = (\sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{j-1})^2$, $j \ge 2$. This operator is symmetric in the space $H = L_2(S)$ and its eigenfunctions $\lambda_n = n(n + l - 2)$, $n \in \mathbb{N}$, have multiplicity $k_{n,l} = (2n + l - 2)(l + n - 3)!((l - 2)!n!)^{-1}$. The eigenfunctions that correspond to these eigenvalues λ_n form the spherical functions $Y_{n,l}^{(k)}(\theta), \ 1 \le k \le k_{n,l}$ ([106], Chapter 13, Section 2). They represent a complete orthonormal system in $L_2(S)$. Because the eigenvalues are positive, δ is a positive definite operator and its spectrum is discrete. We order the spherical functions $Y_{n,l}^{(k)}$ as follows. It is assumed that $l \geq 2$. If $1 \leq k \leq k_{1,l} = l$, then take $\lambda_k = l(l-1)$; $\varphi_k(\theta) = Y_{1,l}^{(k)}(\theta)$, and if $k_{1,l} + \cdots + k_{j,l} < k \leq k_{1,l} + \cdots + k_{j+1,l}$, then $\lambda_k = (j+1)(j+l-1)$; $\varphi_k(\theta) = Y_{j+1,l}^{(k-(k_{1,l}+\cdots+k_{j,l}}(\theta)$. If we substitute these λ_k and φ_k into (1.4.7), we obtain the sequence for an approximate solution of the equation $\delta_n \operatorname{orb}(\delta, u) = \operatorname{orb}_n(\delta, f)$, where δ_n is orbital operator corresponding to δ. For the sequence $\{\operatorname{orb}_n(\delta, u_m)\}$, Theorem 1.4.1 is valid in the space $D(\delta^n)$ with norm (1.4.2), in which A is replaced by δ, and the sequence $\{\operatorname{orb}_n(\delta, u_m)\}$ is convergent to the generalized solution $\operatorname{orb}_n(\delta, u_0)$.

1.5 Linear spline algorithm in space $D(K^{-n})$

Let H be an infinite-dimensional (real or complex) separable Hilbert space with the norm $\|\cdot\|$ generated with the inner product $(\cdot,\cdot)$, and let $K : H \to H$ be a compact, injective, selfadjoint, positive operator. In what follows, we denote by K^{-1} the inverse of K on the image $K(H)$. The operator K^{-1} is not continuous. By an n-orbit of K^{-1} at the point x we mean a finite sequence $\operatorname{orb}_n(K^{-1}, x) := (x, K^{-1}x, \ldots, K^{-n}x)$, $n \in \mathbb{N}_0$. We denote by $D(K^{-n})$ the space of points $x \in H$ to which the operator K^{-1} is applied n-times. Each orbit $\operatorname{orb}_n(K^{-1}, x)$ is uniquely determined by the element $x \in H$, which we call the generating element of this orbit. The space $D(K^{-n})$ is identified with the space of n-orbits of the operator K^{-1}. For simplicity, sometimes we will denote by $\|x\|_n$, instead of $\|\operatorname{orb}_n(K^{-1}, x)\|_n$, the norm of the element $\operatorname{orb}_n(K^{-1}, x) \in D(K^{-n})$, which is generated by the element $x \in H$. We hope that this identification will not lead to misunderstanding. It is obvious that $D(K^{-n})$ is a subspace of H^{n+1}.

Denote by $\{\varphi_k\}$ an orthogonal sequence of eigenfunctions of the operator K with the corresponding decreasing sequence of eigenvalues $\{\lambda_k\}$, $k \in \mathbb{N}$. It is easy to verify that $\{\varphi_k\}$ is a complete system in H. Then K has the form $Ku = \sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi_k)^{-1} (u, \varphi_k)\varphi_k$, where $\lambda_k > 0$ and $\lambda_k \to 0$, if $k \to \infty$.

The left inverse K^{-1} to the operator K is selfadjoint and has the form

$$K^{-1}x = \sum_{k=1}^{\infty} \lambda_k^{-1}(x, \varphi_k)(\varphi_k, \varphi_k)^{-1}\varphi_k \,.$$

The sequence λ_k^{-1} is unbounded and tends to infinity. Therefore, the selfadjoint operator K^{-1} has a discrete spectrum and is positive definite.

We can turn the set $D(K^{-n})$ into the pre-Hilbert space with the help of the following inner product

$$\langle x, y\rangle_n = (x, y) + (K^{-1}x, K^{-1}y) + \cdots + (K^{-n}x, K^{-n}y), \quad n \in \mathbb{N}_0. \qquad (1.5.1)$$

According to (1.5.1), the norm of an element $x \in D(K^{-n})$ has the form

$$\|x\|_n = (\|x\|^2 + \|K^{-1}x\|^2 + \cdots + \|K^{-n}x\|^2)^{1/2}, \quad n \in \mathbb{N}_0. \qquad (1.5.2)$$

It is easy to verify that, since the operator K^{-1} is closed, $D(K^{-n})$ is a Hilbert space.

Let us consider the equation

$$Ku = f. \qquad (1.5.3)$$

This equation in the space $D(K^{-n})$takes the form

$$K_n(u, K^{-1}u, \dots, K^{-n}u) = (f, K^{-1}f, \dots, K^{-n}f)$$

or

$$K_n(\mathrm{orb}_n(K^{-1}, u)) = \mathrm{orb}_n(K^{-1}, f), \qquad (1.5.4)$$

where the operator $K_n : H^{n+1} \to H^{n+1}$ is defined by the equality

$$K_n(u, K^{-1}u, \dots, K^{-n}u) = (Ku, u, K^{-1}u, \dots, K^{-n+1}u),$$

i.e.,

$$K_n(\mathrm{orb}_n(K^{-1}, u)) = \mathrm{orb}_n(K^{-1}, Ku).$$

We call K_n an n-orbital operator corresponding to the operator K and call (1.5.4) an n-orbital equation.

The operator K_n is symmetric and positive in the space $D(K^{-n})$. For arbitrary $n \in \mathbb{N}_0$ in the space $D(K^{-n})$ we have

$$\begin{aligned}\langle K_n u, v\rangle_n &= \big\langle (Ku, u, K^{-1}u, \dots, K^{-n+1}u), (v, K^{-1}v, \dots, K^{-n}v)\big\rangle_n \\ &= (Ku, v) + (u, K^{-1}v) + \cdots + (K^{-n+1}u, K^{-n}v) = \langle u, K_n v\rangle_n\end{aligned}$$

and

$$\langle K_n u, u\rangle_n = (Ku, u) + (u, K^{-1}u) + \cdots + (K^{-n+1}u, K^{-n}u) \geq 0,$$

since all terms are positive.

Let us verify that K_n^{-1} is a symmetric and positive operator in the space $D(K^{-n})$. The symmetry follows from the equality

$$\begin{aligned}\langle K_n^{-1}x, y\rangle_n &= (K^{-1}x, y) + (K^{-2}x, K^{-1}y) + \cdots + (K^{-n-1}x, K^{-n}y) \\ &= \langle x, K_n^{-1}y\rangle_n.\end{aligned}$$

Taking into account the positive definiteness of K^{-1} we get

$$\begin{aligned}\langle K_n^{-1}x, x\rangle_n &= (K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n-1}x, K^{-n}x)\\ &\geq C(x, x) + C(K^{-1}x, K^{-1}x) + \cdots + C(K^{-n-1}x, K^{-n}x)\\ &= C\langle x, x\rangle_n.\end{aligned}$$

Therefore,

$$K_n\varphi_k = K_n(\mathrm{orb}_n(K^{-1}, \varphi_k)) = \mathrm{orb}_n(K^{-1}, K\varphi_k) = \lambda_k \mathrm{orb}_n(K^{-1}, \varphi_k).$$

This means that $\{\mathrm{orb}_n(K^{-1}, \varphi_k)\}$ is an orthogonal sequence of eigenvectors of K_n in the space $D(K^{-n})$. Besides, each eigenvector $\mathrm{orb}_n(K^{-1}, \varphi_k)$ corresponds to the eigennumber λ_k of the operator K_n. The sequence $\{\mathrm{orb}_n(K^{-1}, \varphi_k)\}$ is a complete system in the space $D(K^{-n})$ and

$$K_n(\mathrm{orb}_n(K^{-1}, u)) = \sum_{k=1}^{\infty} \lambda_k(\varphi_k, \varphi_k)_n^{-1}(u, \varphi_k)_n \mathrm{orb}_n(K^{-1}, \varphi_k).$$

Furthermore,

$$\begin{aligned}\langle u, \varphi_k\rangle_n &= (u, \varphi_k) + (K^{-1}u, K^{-1}\varphi_k) + \cdots + (K^{-n}u, K^{-n}\varphi_k)\\ &= (u, \varphi_k) + (u, K^{-2}\varphi_k) + \cdots + (u, K^{-2n}\varphi_k)\\ &= (u, \varphi_k) + \lambda_k^{-2}(u, \varphi_k) + \cdots + \lambda_k^{-2n}(u, \varphi_k)\\ &= (u, \varphi_k)(1 + \lambda_k^{-2} + \cdots + \lambda_k^{-2n}),\end{aligned}$$

and, therefore,

$$\frac{\langle u, \varphi_k\rangle_n}{\langle \varphi_k, \varphi_k\rangle_n} = \frac{(1 + \lambda_k^{-2} + \cdots + \lambda_k^{-2n})(u, \varphi_k)}{(1 + \lambda_k^{-2} + \cdots + \lambda_k^{-2n})(\varphi_k, \varphi_k)} = \frac{(u, \varphi_k)}{(\varphi_k, \varphi_k)}.$$

With this ratio, the last decomposition is rewritten as

$$K_n(\mathrm{orb}_n(K^{-1}, u)) = \sum_{k=1}^{\infty} \lambda_k(\varphi_k, \varphi_k)^{-1}(u, \varphi_k)\mathrm{orb}_n(K^{-1}, \varphi_k). \qquad (1.5.5)$$

In terms of coordinates this means that

$$K^{-i+1}u = \sum_{k=1}^{\infty} \lambda_k^{-i+1}(\varphi_k, \varphi_k)^{-1}(u, \varphi_k)\varphi_k, \quad 0 \leq i \leq n.$$

Our goal is to construct a spline algorithm for the approximate solution of the n-orbital equation (1.5.4) in the space $D(K^{-n})$. For the construction of an approximate solution $U(f)$ we apply the following information about the problem element

f. Let $y = I(f)$, $f \in D(K^{-n})$, be a nonadaptive information of cardinality m, i.e.,

$$y = I(f) = [L_1(f), \dots, L_m(f)], \tag{1.5.6}$$

where $L_i(f) = (f, \varphi_i)_n$, $i = 1, \dots, m$.

Let us construct an interpolatory $y \in I(D(K^{-n}))$ spline in the space $D(K^{-n})$. For this we consider the following spaces: the linear space F_1 consisting of elements of the space $D(K^{-n})$, and $G = D(K^{-n})$ with norm (1.5.2). Let T be the algebraic projection of F_1 on $X = \operatorname{Ker} \mu_F^{\perp}$, i.e., let T be an identical operator from F_1 on $D(K^{-n})$ and $X = (D(K^{-n}), \|\cdot\|_n)$. The solution operator $S_n = K_n^{-1}$ of equation (1.5.4) is defined by the equality

$$S_n(\mathrm{orb}_n(K^{-1}, x)) = \mathrm{orb}_n(K^{-1}, K^{-1}x).$$

The set of problem elements is $F = \{f \in F_1;\ \|T(f)\|_n \leq 1\}$. The spline σ_m interpolatory $y = I(f)$ is defined by the equalities

$$\sigma_m(f) = y \quad \text{and} \quad \|T(\sigma_m)\|_n = \inf\{\|T(z)\|_n,\ z \in D(K^{-n}),\ I(z) = y)\}.$$

As we noted above, the spline $\mathrm{orb}_n(K^{-1}, \sigma_m)$ is the best approximation element of $\mathrm{orb}_n(K^{-1}, f) \in D(K^{-n})$ in the orthogonal complement subspace

$$\operatorname{Ker} I^{\perp} = \operatorname{span}\{\mathrm{orb}_n(K^{-1}, \varphi_1), \dots, \mathrm{orb}_n(K^{-1}, \varphi_m)\} \subset D(K^{-n})$$

with respect to the norm $\|\cdot\|_n$. Therefore, the spline $\mathrm{orb}_n(K^{-1}, \sigma_m)$ interpolatory $y \in I(D(K^{-n}))$ in the space $D(K^{-n})$ has the form

$$\begin{aligned} \mathrm{orb}_n(K^{-1}, \sigma_m) &= \sum_{k=1}^{m} \frac{(f, \varphi_k)_n}{(\varphi_k, \varphi_k)_n} \mathrm{orb}_n(K^{-1}, \varphi_k) \\ &= \sum_{k=1}^{m} \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)} \mathrm{orb}_n(K^{-1}, \varphi_k). \end{aligned} \tag{1.5.7}$$

The coefficient in (1.5.7) does not depend on n. This means that the element $\sigma_m \in D(A^n)$ constructed according to equality (1.5.7) is a spline that simultaneously corresponds to the information $y = [(f, h_0)_n, \dots, (f, h_m)_n]$, as well as to the information $y = [(f, h_0), \dots, (f, h_m)]$.

Using the equality $K_n^{-1}\varphi_k = \lambda_k^{-1}\mathrm{orb}_n(K^{-1}, \varphi_k)$, we obtain

$$\mathrm{orb}_n(K^{-1}, u_m) = S_n \mathrm{orb}_n(K^{-1}, \sigma_m) = \sum_{k=1}^{m} \frac{(f, \varphi_k)}{\lambda_k(\varphi_k, \varphi_k)} \mathrm{orb}_n(K^{-1}, \varphi_k). \tag{1.5.8}$$

This means that

$$\mathrm{orb}_n(K^{-1}, u_m) = S_n \mathrm{orb}_n(K^{-1}, \sigma_m)$$

is a spline algorithm for information (1.5.6), where $L_i(f) = (f, \varphi_i)_n$. It is well known that if equation (1.5.4) admits a generalized solution $\mathrm{orb}_n(K^{-1}, u_0) \in D(K^{-n})$, then the sequence of approximate solutions $\{\mathrm{orb}_n(K^{-1}, u_m)\}$ converges to $\mathrm{orb}_n(K^{-1}, u_0)$ in the space $D(K^{-n})$.

We have proved that in the above notation the following statement is valid.

Theorem 1.5.1. *Let H be a Hilbert space and let K be a compact, injective self-adjoint, positive operator in H. We require that the set $T(\mathrm{Ker}\, I)$ is closed. Then algorithm* (1.5.8) *is a linear spline algorithm for the approximate solution of the n-orbital equation* (1.5.4) *in the space of n-orbits $D(K^{-n})$. Besides, if in the space $D(K^{-n})$ there exists a generalized solution $\mathrm{orb}_n(K^{-1}, u_0)$, then the sequence of approximative solutions $\{\mathrm{orb}_n(K^{-1}, u_m)\}$ converges to $\mathrm{orb}_n(K^{-1}, u_0)$ in the space $D(K^{-n})$.*

Note that the condition in Theorem 1.5.1 on the closedness of the set $T(\mathrm{Ker}\, I)$ is satisfied for the case of information of the form (1.5.6). This follows from the fact that the information functionals are generated by the elements of a basis and are continuous for any $n \in \mathbb{N}$.

Remark 1.5.1. According to Theorem 1.2.5, if the radii of information is finite, i.e., $r(I) < \infty$, then the spline algorithm is central in the worst-case setting. We are interested in finding algorithms having a finite error for ill-posed problems by using incomplete information (1.5.6). The error is being measured in the worst-case, average-case or probabilistic-case setting (see [158], Chapters 6 and 8, for a detailed discussion of the last settings). Algorithms having finite error for a given setting exist if and only if the solution operator S is bounded in that setting. This holds for both the worst-case and average-case setting. In the worst-case setting, this means that there is no algorithm for solving an ill-posed problem whose error is finite, because the solution operator S is unbounded. In the average-case setting, this means that the finite-error algorithms exist if and only if the solution operator is bounded on the average. It was also shown that if the measure is Gaussian and the linear operator S is measurable, then a linear problem is unsolvable on the average if and only if it is ill-posed on the average. It was proved that linear ill-posed problems are solvable on the average for all Gaussian measures. This suggested the following question: is every linear problem well-posed on the average for any Gaussian measure? This question was answered in the affirmative independently in [79, 172]. To prove this, both the notion of zero-mean Gaussian measure and the covariance operator from [175] were essentially used. This is an instance of a general property, as stated in the following theorem from [79]: *linear ill-posed problems are solvable on the average for all Gaussian measures.*

The problem of existence of average-case optimal algorithms for the solution operator was considered in [79] (see also [152], Proposition 4.1). The problem of existence of worst-case optimal algorithms for the solution operator in the Hilbert space of finite n-orbits and in the Fréchet space of all orbits was considered in [170].

We now give several examples of self-adjoint and positive definite operators in a Hilbert space for which the operator K satisfies the conditions of Theorem 1.5.1. Consider the self-adjoint and positive operator K in $L^2(-\infty,\infty)$ that has the form

$$K(u)=\sum_{k=1}^{\infty}(2k+1)^{-1}(u,\varphi_k)\varphi_k.$$

For this operator K, in the space $D(K^{-n})$ consider the corresponding orbital equation $K_n \operatorname{orb}_n(K^{-1},u) = \operatorname{orb}_n(K^{-1},f)$. For a spline σ_m interpolatory y, the spline algorithm has the form

$$\operatorname{orb}_n(K^{-1},u_m)=S_n \operatorname{orb}_n(K^{-1},\sigma_m)=\sum_{k=1}^{m}(2k+1)(f,\varphi_k)\operatorname{orb}_n(K^{-1},\varphi_k),$$

where $S_n=K_n^{-1}$ is the solution operator. According to Theorem 1.5.1, this algorithm is linear and spline in the space $D(K^{-n})$.

2. Integral equations of the first kind.

2.1. Consider the following equation of the first kind (Examples 2.1–2.3 discussed below are compiled in accordance with Examples 2.1, 2.6 and 2.11 of the second chapter of the third part of [77])

$$K(u)=\int_a^b K(s,t)u(s)ds=f(t), \tag{1.5.9}$$

Where

$$K(s,t)=\begin{cases}(s-a)(t-b)(a-b)^{-1}, & a\le s\le t\le b,\\ (t-a)(s-b)(a-b)^{-1}, & a\le t\le s\le b.\end{cases}$$

It is well known that $K(s,t)$ is the Green function for the symmetric and positive definite operator $A=-d^2/dt^2$ in the Hilbert space $L^2[a,b]$ with boundary conditions $u(a)=u(b)=0$. $D(A)$ is a set of functions having absolutely continuous first-order derivatives and quadratically summable second-order derivatives on [a,b]. $D(A^n)$ consists of functions that have on [a,b] absolutely continuous derivatives of order $2n-1$ and quadratically summable derivatives of order $2n$. This space coincides with the Sobolev space $W^{2n}[a,b]$ of order $2n$. The eigenvalues and their corresponding eigenfunctions for A are $\lambda_k = k^2\pi^2/(b-a)^2$ and $\varphi_k(t) = \sqrt{\frac{2}{b-a}} \sin \frac{\pi k(t-a)}{b-a}$, $k \in \mathbb{N}$. Approximate solution of the equation $Ku = f$ in space $D(K^{-n}) = W^{2n}[a,b]$ has the following form

$$\mathrm{orb}_n(K^{-1}, u_m(t)) = \sum_{k=1}^{m} \frac{2k^2\pi^2}{(b-a)^3} \int_a^b f(s) \sin \frac{\pi k(s-a)}{b-a} ds \times \mathrm{orb}_n\left(K^{-1}, \sin \frac{\pi k(t-a)}{b-a}\right).$$

The sequence $\{\mathrm{orb}_n(K^{-1}, u_m)\}$ converges in the space $D(K^{-n})$ to the solution of the equation (1.5.9). For this sequence, the above reasoning applies and according to Theorem 1.5.1, this spline algorithm is linear.

2.2. Consider the integral equation of the first kind (1.5.9), where

$$K(s,t) = \begin{cases} (e^s + e^{2a-s})(e^t + e^{2b-t})2^{-1}(e^{2b} - e^{2a})^{-1}, & a \le s \le t \le b, \\ (e^t + e^{2a-t})(e^s + e^{2b-s})2^{-1}(e^{2b} - e^{2a})^{-1}, & a \le t \le s \le b. \end{cases}$$

It is well known that $K(s,t)$ is the Green function for the symmetric and positive operator $Au = -d^2u/dt^2 + u$ in the Hilbert space $L^2[a,b]$ with the boundary condition $u'(a) = u'(b) = 0$. $D(A)$ is the set of functions having absolutely continuous first-order derivatives and quadratically summable second-order derivatives on [a,b]. $D(A^n)$ consists of functions that have on [a,b] absolutely continuous derivatives of order $2n-1$ and quadratically summable derivatives of order $2n$. This space coincides with the Sobolev space $W^{2n}[a,b]$ of order $2n$. The eigenvalues and their corresponding eigenfunctions for A are $\lambda_k = 1 + k^2\pi^2/(b-a)^2$ and $\mathrm{orb}_n(K^{-1}, \varphi_k(t)) = \sqrt{\frac{2}{b-a}} \cos \frac{\pi k(t-a)}{b-a}$, $k \in \mathbb{N}$. The approximate solution of the equation (1.5.9) has the following form

$$\mathrm{orb}_n(K^{-1}, u_m(t)) = \sum_{k=1}^{m} \left(1 + \frac{k^2\pi^2}{(b-a)^2}\right) \frac{2}{b-a} \int_a^b f(s) \cos \frac{\pi k(s-a)}{b-a} ds \times \mathrm{orb}_n\left(K^{-1}, \cos \frac{\pi k(t-a)}{b-a}\right).$$

The sequence $\{\mathrm{orb}_n(K^{-1}, u_m)\}$ converges in the space $D(K^{-n})$ to the solution of the equation (1.5.9). For this sequence, the above reasoning applies and according to Theorem 1.5.1, this spline algorithm is linear.

2.3. Consider the integral equation (1.5.9), where $a = -\infty,\ b = +\infty$ and

$$K(s,t) = \begin{cases} -\pi^{-1/2} I(-\infty, s) I(t, \infty) \exp \frac{s^2+t^2}{2}, & s \le t, \\ -\pi^{-1/2} I(s, \infty) I(-\infty, t) \exp \frac{s^2+t^2}{2}, & s \ge t, \end{cases}$$

where $I(u,v) = \int_u^v e^{-t^2} dt$. It is well known that $K(s,t)$ is the Green function for symmetric and positive degenerate hypergeometric operator $Au(t) = -d^2u/dt^2 + (t^2+1)u$ in the Hilbert space $L^2[a,b]$ with the boundary condition $u(-\infty) = u(\infty) = 0$. $D(A)$ consists of functions that have absolutely continuous derivatives of the first order and quadratically summable derivatives of order $2n$ on $]-\infty, \infty[$. The eigenvalues and their corresponding eigenfunctions for K are $\lambda_k = 2k$ and $\varphi_k(t),\ k \in \mathbb{N}$. Using the functions φ_k we construct the following approximate solution for the equation (1.5.9)

$$\operatorname{orb}_n(K^{-1}, u_m(t)) = 2 \sum_{k=1}^{m} k \int_{-\infty}^{\infty} f(s)\varphi_k(s)ds \ \operatorname{orb}_n(K^{-1}, \varphi_k(t)).$$

The sequence $\{\operatorname{orb}_n(K^{-1}, u_m)\}$ converges in the space $D(K^{-n})$ to the solution of the equation (1.5.9). For this sequence, the above reasoning applies and according to Theorem 1.5.1, this spline algorithm is linear.

1.6 A linear spline algorithm in the space $D((A^*A)^{-n})$ for the operator A admitting singular value decomposition (SVD)

Let H and M be the Hilbert spaces and let $\{\varphi_k\}$ and $\{\psi_k\}$ be the orthonormal systems in H and M, respectively. For simplicity, for the inner product in H and M, we apply the same notation $(\cdot,\cdot)$.

Further, let A be an operator acting from H to M and having an SVD with respect to the orthonormal systems $\{\varphi_k\}$ and $\{\psi_k\}$ (see [113], Ch. IV, Sect. 1), i.e.,

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, \varphi_k)\psi_k\ ,\ u \in H,\ \lambda_k > 0. \tag{1.6.1}$$

If the operator A has an SVD, it is also said that $\{\varphi_k, \psi_k, \lambda_k\}$, $k \in \mathbb{N}$, represents a singular system for A. The numbers λ_k are called singular numbers of the operator A. Although, in the definition of singular decomposition (1.6.1), the systems $\{\varphi_k\}$ and $\{\psi_k\}$ are required to be orthonormal. This decomposition can also be applied for the cases of orthogonal systems too. Indeed, in this case, the decomposition for Au can again be written in the form

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, \varphi_k)\psi_k = \sum_{k=1}^{\infty} \lambda_k \|\varphi_k\| \cdot \|\psi_k\| \Big(u, \frac{\varphi_k}{\|\varphi_k\|}\Big) \frac{\psi_k}{\|\psi_k\|}. \tag{1.6.2}$$

In (1.6.2), the role of a singular system will be played the triple

$$\Big\{\frac{\varphi_k}{\|\varphi_k\|}, \frac{\psi_k}{\|\psi_k\|}, \lambda_k\|\varphi_k\| \cdot \|\psi_k\|\Big\}.$$

In this case we will say that the operator A admits an SVD (1.6.2) with respect to the orthogonal sequences $\{\varphi_k\}$ and $\{\psi_k\}$ and the triple $\{\varphi_k, \psi_k, \lambda_k\}$, $k \in \mathbb{N}$, represents an orthogonal singular system for A. In general, such operators are not compact, selfadjoint and $\mathrm{Im}A \neq M$. The operator equation

$$Au = f, \tag{1.6.3}$$

in general, is ill-posed and we seek a generalized solution of (1.6.3) in the sense of Moore-Penrose (see [113, Ch. IV]). This means that if $f \in \mathrm{Im}A + \mathrm{Im}A^{\perp}$, we seek a generalized solution that satisfies the equation

$$A^*Au = A^*f. \tag{1.6.4}$$

This solution belongs to the set $(\mathrm{Ker}\, A)^{\perp} = \overline{\mathrm{Im}A^*}$, where $A^* : M \to H$ is the adjoint to the operator A operator in the sense of Hilbert spaces. It follows from (1.6.1) that

$$A^*f = \sum_{k=1}^{\infty} \lambda_k (f, \psi_k)\varphi_k. \tag{1.6.5}$$

The operator $A^*A : H \to H$ has the form

$$A^*Au = \sum_{k=1}^{\infty} \lambda_k^2 (u, \varphi_k)(\psi_k, \psi_k)\varphi_k,\ u \in H. \tag{1.6.6}$$

From (1.6.2) (1.6.5) and (1.6.6) we obtain

$$\begin{aligned} A\varphi_k &= \lambda_k(\varphi_k, \varphi_k)\psi_k,\ A^*\psi_k = \lambda_k(\psi_k, \psi_k)\varphi_k, \\ A^*A\varphi_k &= \lambda_k^2(\varphi_k, \varphi_k)(\psi_k, \psi_k)\varphi_k. \end{aligned} \tag{1.6.7}$$

According to [113], we obtain that if A possesses an SVD (1.6.2), then the unique solution $u^{\dagger}$ of (1.6.3), in the Moore–Penrose sense, is given by the formula

$$u^{\dagger} = \sum_{k=1}^{\infty} \lambda_k^{-1} (\|\varphi_k\| \cdot \|\psi_k\|)^{-2} (f, \psi_k)\varphi_k\ . \tag{1.6.8}$$

The operator A^*A is symmetric and positive. The positiveness follows from the formula

$$(A^*Au, u) = (Au, Au) \geq 0.$$

Also suppose that A is an injective operator. It is clear from the equality $\overline{\mathrm{Ker}\, A^{\perp}} = \overline{\mathrm{Im} A^*}$ that under the above conditions, the equality $\overline{\mathrm{Im} A^* A} = H$ is valid. This means that the operator A^*A is selfadjoint, having positive eigenvalues $\lambda_k^2(\varphi_k, \varphi_k)(\psi_k, \psi_k)$, which correspond to the eigenelements φ_k.

If the systems $\{\varphi_k\}$ and $\{\psi_k\}$ in (1.6.1) are orthonormal and $\lambda_k \to 0$, then the operator A is compact (see [41], Chapter 1, Section 2). From this it follows that if these systems are only orthogonal and

$$\lim_{k\to\infty} \lambda_k \|\varphi_k\| \cdot \|\psi_k\| = 0 \,,$$

then A is compact. Then A^*A is also compact and we can apply the results of Section 1.5 for the operator $K := A^*A : H \to H$. According to formulas (1.5.8) and (1.6.6), an approximate solution u_m of (1.6.4) has the form

$$\begin{aligned} u_m &= \sum_{k=1}^{m} \lambda_k^{-2} \|\varphi_k\|^{-4} \|\psi_k\|^{-2} (A^* f, \varphi_k)\varphi_k \\ &= \sum_{k=1}^{m} \lambda_k^{-2} \|\varphi_k\|^{-4} \|\psi_k\|^{-2} (f, A\varphi_k)\varphi_k \\ &= \sum_{k=1}^{m} \lambda_k^{-1} (\|\psi_k\| \|\varphi_k\|)^{-2} (f, \psi_k)\varphi_k \,. \end{aligned} \tag{1.6.9}$$

This means that the approximate solution u_m of equation (1.6.4) coincides with the m-th partial sum of the generalized solution in the sense of Moore–Penrose represented by (1.6.8). Thus (1.6.9) is the truncated SVD for the regularization method (see [96], Theorem 1.2)

$$T_\gamma f = \sum_{k=1}^{\infty} F_\gamma(\lambda_k) (\|\psi_k\| \cdot \|\varphi_k\|)^{-2} (f, \psi_k)\varphi_k$$

with the filter

$$F_\gamma(\lambda_k) = \begin{cases} \lambda_k^{-1} & \text{if } \ k \le \frac{1}{\gamma}, \\ 0 & \text{if } \ k > \frac{1}{\gamma}. \end{cases}$$

In other words, if we take $\gamma = \frac{1}{m}$, then for such a filter we have $T_\gamma f = T_{1/m} f = \sum_{k\le m} \lambda_k^{-1} (\|\psi_k\| \cdot \|\varphi_k\|)^{-2} (f, \psi_k)\varphi_k = u_m$.

We consider equation (1.6.4) in the space $D((A^*A)^{-n})$. Let us apply Theorem 1.5.1 for the case $K = A^*A$, where A admits an SVD (1.6.2). The norm (1.5.2), which is generated by the inner products (1.5.1), for $K = A^*A$ has the form

$$\begin{aligned} \langle x, y\rangle_n = (x, y) &+ \big((A^*A)^{-1}x, (A^*A)^{-1}y\big) \\ &+ \cdots + \big((A^*A)^{-n}x, (A^*A)^{-n}y\big) \,, \quad n \in \mathbb{N}_0, \end{aligned}$$

and

$$\|x\|_n = \left(\|x\|^2 + \|(A^*A)^{-1}x\|^2 + \cdots + \|(A^*A)^{-n}x\|^2\right)^{1/2}, \quad n \in \mathbb{N}_0. \tag{1.6.10}$$

The space $D((A^*A)^{-n})$ is isomorphic to the space of n-orbits $\mathrm{orb}_n((A^*A)^{-1}, x)$ of the operator $(A^*A)^{-1}$. This isomorphism is obtained by the mapping

$$\begin{aligned} D((A^*A)^{-n}) \ni x \to\ & \mathrm{orb}_n((A^*A)^{-1}, x) \\ &= \left(x, (A^*A)^{-1}x, \ldots, (A^*A)^{-n}x\right). \end{aligned} \tag{1.6.11}$$

In the sequel, following our previous agreements, sometimes instead of the norm $\|\mathrm{orb}_n((A^*A)^{-1}, x)\|_n$ of the element $\mathrm{orb}_n((A^*A)^{-1}, x) \in D((A^*A)^{-n})$ generated by $x \in H$ we simply write $\|x\|_n$. If some operator B acts on $D((A^*A)^{-n})$, instead of $B(\mathrm{orb}_n((A^*A)^{-1}, x))$ we will simply write Bx.

Equation (1.6.4) in the space $D((A^*A)^{-n})$ actually has the form

$$(A^*A)_n(\mathrm{orb}_n((A^*A)^{-1}, u)) = \mathrm{orb}_n((A^*A)^{-1}, A^*f), \tag{1.6.12}$$

where

$$(A^*A)_n(\mathrm{orb}_n((A^*A)^{-1}, u)) = (A^*Au, u, \ldots, (A^*A)^{-n+1}u)$$

is an n-orbital operator for A^*A and

$$\mathrm{orb}_n((A^*A)^{-1}, A^*f) = (A^*f, (A^*A)^{-1}A^*f, \ldots, (A^*A)^{-n}A^*f).$$

The decomposition of the element $\mathrm{orb}_n((A^*A)^{-1}, u)$ with respect to the system $\mathrm{orb}_n((A^*A)^{-1}, \varphi_k)$ in the space $D((A^*A)^{-n})$ has the form

$$\begin{aligned} \mathrm{orb}_n((A^*A)^{-1}, u) &= \sum_{k=1}^{\infty}(u, \varphi_k)_n(\varphi_k, \varphi_k)_n^{-1}\mathrm{orb}_n((A^*A)^{-1}, \varphi_k) \\ &= \sum_{k=1}^{\infty}(u, \varphi_k)(\varphi_k, \varphi_k)^{-1}\mathrm{orb}_n((A^*A)^{-1}, \varphi_k). \end{aligned}$$

Therefore, according to (1.6.7), the orbital operator $(A^*A)_n$ has the form

$$\begin{aligned} &(A^*A)_n\,\mathrm{orb}_n((A^*A)^{-1}, u) \\ &\quad= \sum_{k=1}^{\infty}(u, \varphi_k)(\varphi_k, \varphi_k)^{-1}(A^*A)_n\mathrm{orb}_n((A^*A)^{-1}, \varphi_k) \\ &\quad= \sum_{k=1}^{\infty}\lambda_k^2(u, \varphi_k)(\psi_k, \psi_k)\mathrm{orb}_n((A^*A)^{-1}, \varphi_k). \end{aligned} \tag{1.6.13}$$

This representation is an analogy of (1.5.5). The latter equality in coordinates is written as

$$(A^*A)^{-i+1}u = \sum_{k=1}^{\infty}(\lambda_k^2(\varphi_k,\varphi_k)(\psi_k,\psi_k))^{-i+1}(u,\varphi_k)(\varphi_k,\varphi_k)^{-1}\varphi_k,\ 0 \le i \le n.$$

In the case of $i = 0$, we obtain equality (1.6.6).

For the inverse $S_n = (A^*A)_n^{-1}$ to the operator $(A^*A)_n$ on the range of $(A^*A)_n$, we have

$$\begin{aligned}(A^*A)_n^{-1}&\big(x,(A^*A)^{-1}x,\ldots,(A^*A)^{-n}x\big)\\ &= \big((A^*A)^{-1}x,(A^*A)^{-2}x,\ldots,(A^*A)^{-n-1}x\big)\end{aligned}$$

and

$$S_n(\mathrm{orb}_n((A^*A)^{-1},x)) = \mathrm{orb}_n((A^*A)^{-1},(A^*A)^{-1}x).$$

It will be noted that S_n is the n-orbital operator corresponding to the operator $(A^*A)^{-1}$.

1.6.1 Construction of a spline and a spline algorithm in the space of n-orbits $D((A^*A)^{-n})$

Our goal is to construct an algorithm for the approximate solution of equation (1.6.4) in the space $D((A^*A)^{-n})$. For the construction of the approximate solution $U(f)$ we apply some information about the right-hand side A^*f of the n-orbital equation (1.6.12). Below we will use the notation of Section 1.1. Let us construct the spline interpolatory $y \in I(D((A^*A)^{-n}))$ in the space $D((A^*A)^{-n})$, where I is nonadaptive information of cardinality m and

$$y = I(A^*f) = \big[(A^*f,\varphi_1),\ldots,(A^*f,\varphi_m)\big].$$

Let F_1 be a linear space consisting of elements of the space $D((A^*A)^{-n})$, let $G = X = D((A^*A)^{-n})$ with norm (1.6.10), and let T be an identical operator acting from F_1 to X. Let the set of problem elements be

$$F = \{f \in D((A^*A)^{-n});\ \|Tf\|_n \le 1\},$$

where

$$\|Tf\|_n = \big(\|f\|^2 + \|(A^*A)^{-1}f\|^2 + \cdots + \|(A^*A)^{-n}f\|^2\big)^{1/2},$$

and let the solution operator be $S_n = (A^*A)_n^{-1}$. The spline σ_m interpolatory $y = I(A^*f)$ in the space $D((A^*A)^{-n})$ is defined by the equalities $I(\sigma_m) = y$ and

$$\|T(\sigma_m)\|_n = \inf\{\|T(z)\|_n :\ z \in D(K^{-n}),\ I(z) = y\}.$$

Applying (1.5.7) we obtain that the spline σ_m in the space $D((A^*A)^{-n})$ has the form

$$\begin{aligned} \mathrm{orb}_n((A^*A)^{-1},\sigma_m) &= \sum_{i=1}^{m} \frac{(A^*f,\varphi_i)}{(\varphi_i,\varphi_i)} \mathrm{orb}_n((A^*A)^{-1},\varphi_i) \\ &= \sum_{i=1}^{m} \frac{(f,A\varphi_i)}{(\varphi_i,\varphi_i)} \mathrm{orb}_n((A^*A)^{-1},\varphi_i) \\ &= \sum_{i=1}^{m} \lambda_i(f,\psi_i)\mathrm{orb}_n((A^*A)^{-1},\varphi_i). \end{aligned} \tag{1.6.14}$$

The spline σ_m is the best approximation element of $A^*f \in D((A^*A)^{-n})$ in the orthogonal complement subspace $\mathrm{Ker}\, I^{\perp} \subset D((A^*A)^{-n})$ with respect to the norm (1.6.10), and the coefficients in (1.6.14) do not depend on n. This means that the element $\sigma_m \in D((A^*A)^{-n})$ constructed according to equality (1.6.14) is a spline that simultaneously corresponds to the information $y = [(A^*f,\varphi_1)_n,\dots,(A^*f,\varphi_m)_n]$ and to the information $y_1 = [(A^*f,\varphi_1),\dots,(A^*f,\varphi_m)]$. It is easy to see that

$$S_n(\mathrm{orb}_n(A^*A)^{-1},\varphi_i) = \lambda_i^{-2}(\varphi_i,\varphi_i)^{-1}(\psi_i,\psi_i)^{-1}\mathrm{orb}_n((A^*A)^{-1},\varphi_i).$$

Also, according to (1.6.9) and (1.6.14), we have

$$\begin{aligned} \mathrm{orb}_n((A^*A)^{-1},u_m) &= \sum_{i=1}^{m} \lambda_i^{-1}(\|\varphi_i\|\cdot\|\psi_i\|)^{-2}(f,\psi_i)\mathrm{orb}_n((A^*A)^{-1},\varphi_i) \\ &= \sum_{i=1}^{m} \lambda_i(f,\psi_i)S_n(\varphi_i) = S_n\left(\sum_{i=1}^{m} \lambda_i(f,\psi_i)\varphi_i\right) \\ &= S_n(\sigma_m). \end{aligned} \tag{1.6.15}$$

Consider the problem of convergence of the sequence $\{\mathrm{orb}_n((A^*A)^{-1},u_m)\}$ defined by (1.6.9) to the element $u^{\dagger}$defined by (1.6.8), when $A^*f \in D((A^*A)^{-n})$. By our assumption, A is an injective operator. Therefore, the operator A^*A is such, too. From this and the symmetry of the operator $(A^*A)^{-1}$, with the help of (1.6.7) and (1.6.9), we can write u_m in the following way:

$$\begin{aligned} u_m &= \sum_{k=1}^{m} \|\varphi_k\|^{-2}(f,A\varphi_k)(A^*A)^{-1}\varphi_k = \sum_{k=1}^{m} \|\varphi_k\|^{-2}(A^*f,\varphi_k)(A^*A)^{-1}\varphi_k \\ &= \sum_{k=1}^{m} \left((A^*A)^{-1}A^*f,\frac{\varphi_k}{\|\varphi_k\|}\right)\frac{\varphi_k}{\|\varphi_k\|}. \end{aligned}$$

Hence

$$
\begin{aligned}
(A^*A)^{-(s-1)}u_m &= \sum_{k=1}^{m} \Big((A^*A)^{-1}A^*f, \frac{\varphi_k}{\|\varphi_k\|}\Big)(A^*A)^{-(s-1)}\frac{\varphi_k}{\|\varphi_k\|} \\
&= \sum_{k=1}^{m} \Big((A^*A)^{-s}A^*f, \frac{\varphi_k}{\|\varphi_k\|}\Big)\frac{\varphi_k}{\|\varphi_k\|} \qquad (1.6.16)
\end{aligned}
$$

for any $1 \le s \le n$. The right-hand side of (1.6.16) is the m-th partial sum of the Fourier series of the element $(A^*A)^{-s}A^*f$ with respect to the orthonormal system $\varphi_k/\|\varphi_k\|$. Since A^*A is a compact self-adjoint operator, its eigenelements φ_k, according to the Hilbert-Schmidt theorem, constitute a dense set in H. Therefore, the sequence $\{(A^*A)^{-(s-1)}u_m\}$ defined by (1.6.16) converges in H to $(A^*A)^{-s}A^*f = (A^*A)^{-s}u^\dagger$ for all $1 \le s \le n$, i.e., the sequence $\{\mathrm{orb}_n((A^*A)^{-1}, u_m)\}$ conoverges to $\mathrm{orb}_n((A^*A)^{-1}, u^\dagger)$ in the space $D((A^*A)^{-n+1})$.

The foregoing and Theorem 1.5.1 imply the following theorem.

Theorem 1.6.1. *Let H and M be Hilbert spaces, let $A : H \to M$ be a compact operator, and let $\{\varphi_k, \psi_k, \lambda_k\}$, $k \in \mathbb{N}$, form an orthogonal singular system for A. Then algorithm* (1.6.15) *is a linear spline in the space $D((A^*A)^{-n})$. Moreover, if $A^*f \in D((A^*A)^{-n})$, then the sequence of the approximate solutions $\{\mathrm{orb}_n((A^*A)^{-1}, u_m)\}$ converges to $\mathrm{orb}_n((A^*A)^{-1}, u^\dagger)$ in the space $D((A^*A)^{-n+1})$.*

Proof. From the compactness of the operator A it follows that A^*A is a compact, selfadjoint and positive operator. This implies that all conditions of Theorem 1.5.1 for the operator $K = A^*A$ in the space $D((A^*A)^{-n})$ are satisfied. □

1.7 A linear spline algorithm of computerized tomography (CT) in the space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$

The main problem of computerized tomography consists in the reconstruction of the function by its integral over hyperplanes. The map $\mathfrak{R}$, which to the function f defined on the p-dimensional Euclidean space $\mathbb{R}^p$ corresponds the integrals of f along all hyperplanes, is called the Radon operator.

We use the standard parametrization of a hyperplane by a normal unit vector ω and its distance s from the origin as follows. The Radon operator $\mathfrak{R}$ maps a density function u to its integrals over all hyperplanes and is defined by the formula

$$
\mathfrak{R}u(\omega, s) = \int\limits_{(t,\omega)=s} u(t)dt = \int\limits_{\omega^\perp} u(s\omega + t)dt, \qquad (1.7.1)
$$

where $\omega \in \mathbb{S}^{p-1} = \{x \in \mathbb{R}^p : \ |x| = 1\}$ $(p > 1)$ and u belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^p)$ of rapidly decreasing in $\mathbb{R}^p$ functions. According to the well-known Schwartz theorem ([113], Section II.1), $\mathfrak{R}$ is the injective operator acting from $\mathcal{S}(\mathbb{R}^p)$ to the Schwartz space $\mathcal{S}(Z)$, where Z is the cylinder $Z = \mathbb{S}^{p-1} \times \mathbb{R}$. Moreover, $\mathfrak{R}$ is a topological monomorphism with the closed countable codimensional range [69].

The Radon operator, which is defined by (1.7.1) only for the functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^p)$, admits a continuous extension to some weighted L_2-space. Let $W_\nu(x) = (1 - |x|^2)^{\nu - p/2}$ be the weight function, defined on the unit ball $\Omega^p = \{x \in \mathbb{R}^p : \ |x| \le 1\}$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^p$. Let $w_\nu(s) = (1 - s^2)^{\nu - 1/2}$, $s \in [-1, 1]$, be the weight function defined on the cylinder Z. It is shown in ([41], p. 12) that $\mathfrak{R}$ is a continuous operator acting from the space $H = L_2(\Omega^p, W_\nu^{-1})$ into the space $M = L_2(Z, w_\nu^{-1})$ which is endowed with the usual norm. If $\nu > \frac{p}{2} - 1$, the operator $\mathfrak{R}$ acting in these spaces admits a SVD with respect to products of Gegenbauer and spherical harmonics, which was obtained by A. Louis [95].

Beforehand, we introduce some notation:

- $P_r^{(\alpha,\beta)}$ is the Jacobi polynomial of degree r and indices α, β.
- C_r^ν is the Gegenbauer polynomial of degree r and index ν.
- Γ is the second-kind Euler integral.
- By
$$\{Y_{lk}, \ k = 1, \dots, N(p, l)\}$$
we denote the orthonormal basis of spherical functions defined on $\mathbb{S}^{p-1}$, where
$$l = 0, 1, \dots, \quad N(p, l) = \frac{(2l + p - 2)(p + l - 3)!}{l!(p - 2)!}, \ \ p \ge 2;$$
- We set
$$v_{rlk}^\nu(x) = W_\nu(x)|x|^l P_{(r-l)/2}^{(\nu - p/2, l + p/2 - 1)}(2|x|^2 - 1) Y_{lk}(x/|x|); \quad (1.7.2)$$
$$u_{rlk}^\nu(\omega, s) = d_{rl} w_\nu(s) C_r^\nu(s) Y_{lk}(\omega),$$
where
$$d_{rl} = \pi^{p/2 - 1} 2^{2\nu - 1} \frac{\Gamma((r - l)/2 + \nu - p/2 + 1)\Gamma(r + 1)\Gamma(\nu)}{\Gamma((r - l)/2 + 1)\Gamma(r + 2\nu)}. \quad (1.7.3)$$

- We set

$$\lambda_{rlk}^2 = \frac{2^{2\nu}\Gamma((r+l)/2+\nu)\Gamma((r-l)/2+\nu-p/2+1)\Gamma(r+1)}{\pi^{1-p}\Gamma((r+l+p)/2)\Gamma((r-l)/2+1)\Gamma(r+2\nu)} = \lambda_{rl}^2. \quad (1.7.4)$$

We note that in notation (1.7.2)–(1.7.4), $P_0^{(\alpha,\beta)} \equiv 1$, $C_0^\lambda \equiv 1$ and $Y_{0k} \equiv 1$.

Proposition 1.7.1 ([41], Proposition 1.3.2)**.** *The system* $\{v_{rlk}^\nu,\ u_{rlk}^\nu,\ \lambda_{rl}\}$*,* $r \geq 0$*,* $0 \leq l \leq r$*,* $k = 1, \dots, N(p,l)$*, where* v_{rlk}^ν*,* u_{rlk} *and* λ_{rl} *are defined by* (1.7.2)–(1.7.4)*, is an orthogonal singular system for the Radon operator* $\mathfrak{R}$ *acting from* $L_2(\Omega^p, W_\nu^{-1})$ *to* $L_2(Z, w_\nu^{-1})$*. In other words,*

$$\mathfrak{R}u(\omega,s) = \sum_{r=0}^{\infty}{\sum_{l\leq r}}' \lambda_{rl} \sum_{k=1}^{N(p,l)} (u, v_{rlk}^\nu)_{L_2(\Omega^p, W_\nu^{-1})} \cdot u_{rlk}^\nu(\omega,s)$$

where Σ' *means that the summability takes place only for even* $r+l$*.*

It should be noted that the systems $\{u_{rlk}^\nu\}$ and $\{v_{rlk}^\nu\}$ are orthogonal, but not orthonormal ([41], p. 13). To obtain decomposition with respect to the orthonormal systems, we rewrite the decomposition from Proposition 1.7.1 as follows:

$$\mathfrak{R}u(\omega,s) = \sum_{r=0}^{\infty}{\sum_{l\leq r}}' \lambda_{rl} \sum_{k=1}^{N(p,l)} \|v_{rlk}^\nu\| \cdot \|u_{rlk}^\nu\| \Big(u, \frac{v_{rlk}^\nu}{\|v_{rlk}^\nu\|}\Big)_{L_2(\Omega^p, W_\nu^{-1})} \cdot \frac{u_{rlk}^\nu(\omega,s)}{\|u_{rlk}^\nu\|}.$$

From here and with the help of (1.6.7) we obtain that the solution in the sense of Moore-Penrose $u^\dagger$ of the equation $\mathfrak{R}u = f$, where $\mathfrak{R} : H \to M$, is given by the formula

$$u^\dagger = \sum_{r=0}^{\infty}{\sum_{l\leq r}}' \lambda_{rl}^{-1} \sum_{k=1}^{N(p,l)} (\|u_{rlk}^\nu\| \cdot \|v_{rlk}^\nu\|)^{-2} (f, u_{rlk}^\nu)_{L_2(Z, w_\nu^{-1})} v_{rlk}^\nu(x),$$
$$x \in \Omega^p. \quad (1.7.5)$$

Proposition 1.7.2 ([168])**.** *If* $\{v_{rlk}^\nu, u_{rlk}^\nu\}$ *and* λ_{rl}*,* $l \leq r$*,* $1 \leq k \leq N(p,l)$*, are represented by* (1.7.2)–(1.7.4)*, then*

$$\lim_{r\to\infty} \lambda_{rl}\|u_{rlk}^\nu\| \cdot \|v_{rlk}^\nu\| = 0.$$

For approximation inversion of the Radon operator $\mathfrak{R}$, i.e., for the computerized tomography problem, we construct a linear spline algorithm for the equation $\mathfrak{R}^*\mathfrak{R}u = \mathfrak{R}^*f$ in the Hilbert space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$ with the norm

$$\|x\|_n = (\|x\|^2 + \|(\mathfrak{R}^*\mathfrak{R})^{-1}x\|^2 + \dots + \|(\mathfrak{R}^*\mathfrak{R})^{-n}x\|^2)^{1/2}. \quad (1.7.6)$$

According to (1.6.12), this n-orbital equation takes the form

$$(\mathfrak{R}^*\mathfrak{R})_n(\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u)) = (\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, \mathfrak{R}^* f)), \tag{1.7.7}$$

where

$$(\mathfrak{R}^*\mathfrak{R})_n(\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u)) = \mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, \mathfrak{R}^*\mathfrak{R}u)$$

is an n-orbital operator for $\mathfrak{R}^*\mathfrak{R}$ and

$$(\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, \mathfrak{R}^* f)) = \{\mathfrak{R}^* f, (\mathfrak{R}^*\mathfrak{R})^{-1}\mathfrak{R}^* f, \dots, (\mathfrak{R}^*\mathfrak{R})^{-n}\mathfrak{R}^* f\}.$$

According to (1.6.13), for (1.7.7) we have the representation

$$\begin{aligned}(\mathfrak{R}^*\mathfrak{R})_n&(\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u)) \\ &= \sum_{r=0}^{\infty} {\sum_{l\le r}}' \lambda_{rl}^2 \sum_{k=1}^{N(p,l)} (u_{rlk}^{\nu}, u_{rlk}^{\nu})(u, v_{rlk}^{\nu}) \mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, v_{rlk}^{\nu}).\end{aligned} \tag{1.7.8}$$

The m-th truncated singular value decomposition (TSVD) corresponding to the solution of (1.7.5) has the form

$$\begin{aligned}&\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u_m) \\ &= \sum_{r=0}^{m} {\sum_{l\le r}}' \lambda_{rl}^{-1} \sum_{k=1}^{N(p,l)} (\|u_{rlk}^{\nu}\| \cdot \|v_{rlk}^{\nu}\|)^{-2} (f, u_{rlk}^{\nu})_{L_2(Z, w_\nu^{-1})} \mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, v_{rlk}^{\nu}), \\ &\qquad x \in \Omega^p,\end{aligned} \tag{1.7.9}$$

where Σ' means that the summability takes place only for even $r + l$.

Below we will use the notation of Section 1.6. Let us construct the spline interpolatory $y \in I(D((\mathfrak{R}^*\mathfrak{R})^{-n}))$ in the space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$, where I is nonadaptive information of cardinality $m + 1$, i.e.,

$$y = I(\mathfrak{R}^* f) = [(\mathfrak{R}^* f, u_{001}^{\nu})_n, \dots, (\mathfrak{R}^* f, u_{mmN(p,m)}^{\nu})_n].$$

Let F_1 be the linear space consisting of elements of the space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$, let $G = X = D((\mathfrak{R}^*\mathfrak{R})^{-n})$ with norm (1.7.6), and let T be the identical operator acting from F_1 to X. Let the set problem element be

$$F = \{f \in D((\mathfrak{R}^*\mathfrak{R})^{-n});\ \|T(f)\|_n \le 1\},$$

where

$$\|T(f)\|_n = (\|f\|^2 + \|(\mathfrak{R}^*\mathfrak{R})^{-1} f\|^2 + \cdots + \|(\mathfrak{R}^*\mathfrak{R})^{-n} f\|^2)^{1/2}.$$

The solution operator $S_n = (\mathfrak{R}^*\mathfrak{R})_n^{-1}$, where $(\mathfrak{R}^*\mathfrak{R})_n$ is defined by (1.7.8). The spline σ_m interpolatory $y = I(f)$ in the space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$ is defined by the equalities $I(\sigma_m) = y$ and

$$\|T(\sigma_m)\|_n = \inf\{\|T(\mathfrak{R}^* z)\|_n;\ z \in D((\mathfrak{R}^*\mathfrak{R})^{-n}),\ I(z) = y\}.$$

Applying (1.5.7) and (1.6.13), we can obtain the spline σ_m in the space $D((\mathfrak{R}^*\mathfrak{R})^{-n})$, but we obtain σ_m from the equality $u_m = S_n(\sigma_m) = \varphi^s(\sigma_m)$. According to (1.6.15) we have

$$\begin{aligned}
&\mathrm{orb_n}((\mathfrak{R}^*\mathfrak{R})^{-1}, \sigma_m(y)) = S_n^{-1}(u_m) = (\mathfrak{R}^*\mathfrak{R})_n u_m \\
&= (\mathfrak{R}^*\mathfrak{R})_n \sum_{r=0}^{m} {\sum_{l\le r}}' \lambda_{rl}^{-1} \sum_{k=1}^{N(p,l)} (\|u_{rlk}^\nu\| \cdot \|v_{rlk}^\nu\|)^{-2} (f, u_{rlk}^\nu)_{L_2(Z, w_\nu^{-1})} v_{rlk}^\nu(x) \\
&= \sum_{r=0}^{m} {\sum_{l\le r}}' \lambda_{rl} \sum_{k=1}^{N(p,l)} (f, u_{rlk}^\nu)_{L_2(Z, w_\nu^{-1})} \mathrm{orb_n}((\mathfrak{R}^*\mathfrak{R})^{-1}, v_{rlk}^\nu(x)),\ x \in \Omega^p.
\end{aligned}$$

If we collect the results of this section and apply Theorem 1.6.1, we obtain the following theorem.

Theorem 1.7.1. *Let* $\{v_{rlk}^\nu, u_{rlk}^\nu, \lambda_{rl}\}$, $l \le r$, $1 \le k \le N(p,l)$, *be an orthogonal singular system for the Radon operator* $\mathfrak{R}$, *which acts from* $L_2(\Omega^p, W_\nu^{-1})$, $\nu > \frac{p}{2} - 1$, *to the space* $L_2(Z, w_\nu^{-1})$. *Then the algorithm*

$$\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u_m) = S_n(\sigma_m) = \varphi^s(I(\mathfrak{R}^* f)),$$

where $\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u_m)$ *is defined according to* (1.7.9)*, is a linear spline for the solution operator* $S_n = (\mathfrak{R}^*\mathfrak{R})_n^{-1}$ *and nonadaptive information*

$$I(\mathfrak{R}^* f) = [(\mathfrak{R}^* f, u_{001}^\nu)_n, \ldots, (\mathfrak{R}^* f, u_{mmN(p,m)}^\nu)_n].$$

Moreover, if $\mathfrak{R}^* f \in D((\mathfrak{R}^*\mathfrak{R})^{-n})$, *then the sequence of approximative solutions* $\{\mathrm{orb}_n((\mathfrak{R}^*\mathfrak{R})^{-1}, u_m)\}$ *converges to the solution of equation* (1.7.7) *(in the sense of Moore–Penrose) in the space* $D((\mathfrak{R}^*\mathfrak{R})^{-n+1})$.

Let us go back to equation (1.7.7). After applying the SVD for the Radon transform, we construct an approximate solution in the form of the TSVD (1.7.9). Let us write the solution obtained by this method in the form $u_m = S_\gamma$, $\gamma = \frac{1}{m}$. According to the well-known result of A. Louis ([96], Theorem 3), we can say that this method can be written as an approximative inverse with mollifier

$$e_\gamma(x,y) = \sum_{r=0}^{m} {\sum_{l\le r}}' \sum_{k=1}^{N(p,l)} \|v_{rlk}^\nu\|^{-2} v_{rlk}^\nu(x) \cdot v_{rlk}^\nu(y), \quad \gamma = \frac{1}{m}, \quad x \in \Omega^p, \quad y \in Z.$$

CHAPTER 2

New results in the theory of locally convex spaces

2.1 Different topologies of uniform convergence on locally convex spaces

Below we will use the following most generally accepted designations and terms, mainly borrowed from [50, 82, 83, 144, 147].

Let $(E, \mathfrak{T})$ be a linear space over a field R or C and $\mathfrak{T}$ be the topology on E. The space $(E, \mathfrak{T})$ is called a linear topological space, if E is linear space and linear operations are continuous in the topology $\mathfrak{T}$. $(E, \mathfrak{T})$ is said to be LCS if there is an absolutely convex (convex and balanced) basis of neighborhoods of the zero $U_0(E)$ in E.

The locally convex topology $\mathfrak{T}$ of the space $(E, \mathfrak{T})$ is generated by a family of seminorms. It is enough to take the Minkowski functionals for neighborhoods $U_\alpha \in U_0(E)$, whose positive multiples form a basis at zero. Note that if M is a bounded and absolutely convex set of LCS E, and μ_M is the Minkowski functional of set M, then μ_M is a norm on the space $E_M = \operatorname{span} M \subset E$. If M is an absolutely convex and absorbing set in E, then μ_M is seminorm on E. Moreover, if M is a neighborhood of zero in E, then μ_M is a continuous seminorm on E. In this chapter, Minkowski functionals for neighborhoods U_α are denoted by μ_{U_α} or via $p_{U_\alpha}(\cdot)$, or via $p_\alpha(\cdot)$. Families $\{U_\alpha\}$ and $\{p_\alpha\}$ are called generating families, respectively, of neighborhoods and semi-norms for the $\mathfrak{T}$ topology. It is known that every element of a generating family of seminorms is a continuous function in the $\mathfrak{T}$ topology. Also, the $\mathfrak{T}$ topology is Hausdorff or separated if and only if for every non-zero $x \in E$ and for every family of seminorms $\mathcal{P} = \{p_\alpha;\ \alpha \in A\}$, generating $\mathfrak{T}$, there is a seminorm $p \in \mathcal{P}$ such that $p(x) > 0$. A subset B of LCS $(E, \mathfrak{T})$ with a basis of neighborhoods of zero $U_0(E)$ is called bounded, if for each neighborhood $U \in U_0(E)$ there exists $\lambda \in R^+$ such that $\lambda B \subset U$ (R^+ denotes the set of positive real numbers). The family of bounded sets $\mathfrak{B}(E)$ in E is called fundamental if every bounded set of the space E is contained in some set of $\mathfrak{B}(E)$.

Theorem 2.1.1 ([81], p. 205)**.** *An LCS $(E, \mathfrak{T})$ is metrizable if and only if it is separated and has a countable basis of neighbourhoods of zero $\{V_n\}$. Topology of metrizable space can be given by a translation-invariant metric.*

It should be noted that if the sequence of the mentioned neighbourhoods $\{V_n\}$ is non-increasing, then the sequence of the corresponding Minkowski functionals $\{p_n(\cdot)\}$ is non-decreasing. Complete, metrizable, locally convex spaces are called Fréchet spaces.

Let E and F be a pair of linear spaces and $\langle\cdot,\cdot\rangle$ bilinear form on $E \times F$ satisfying the axioms:

(S_1) if $\langle x_0, y\rangle = 0$ for all $y \in F$, then $x_0 = 0$,

(S_2) if $\langle x, y_0\rangle = 0$ for all $x \in E$, then $y_0 = 0$.

The triple $(E, F, \langle\cdot,\cdot\rangle)$ is called a dual system or duality and is denoted by the symbol $\langle E, F\rangle$. It is obvious that $\langle E, E'\rangle$ and $\langle E, E^*\rangle$ are dual systems, where E' is the space of all continuous linear functionals on E and E^* is the space of all linear functionals on E. From the symmetry of $\langle E, F\rangle$ regarding E and F, any sentence regarding E can also be expressed relative to F by a simple variable fields E and F. Let $\langle E, F\rangle$ be dual. For each $M \subset E$ a set defined

$$M^0 = \{y \in F;\ |\langle x, y\rangle| \le 1 \text{ for all } x \in M\}$$

is called the polar of the set M in F. Sometimes instead of M^0 we will write M^{0F}. If M is a subspace in E, then its polar is weakly closed ($\sigma(F, E)$-closed) subspace in F, denoted by $M^\perp$ and called *annulator* for subspace M.

Let $(E, \mathfrak{T})$ be an LCS. A set $M \subset E'$ is said to be equicontinuous if there exists a $\mathfrak{T}$-neighborhood V of zero in E such that $M \subset V^0$. It should be noted that polars taken with respect to the system $\langle E, E'\rangle$ of any fundamental families of equicontinuous sets in E' form basis of neighborhoods of zero in E, i.e. topology $\mathfrak{T}$ is topology uniform convergence on a family of equicontinuous sets of the dual space E'.

Let $\langle E, F\rangle$ be dual system and $\mathfrak{M}$ be a saturated family of $\sigma(F, E)$-bounded subsets of F. The family $\mathfrak{M}$ is called saturated if the following three conditions are satisfied: 1) If $M \in \mathfrak{M}$, so does every subset of M; 2) If $M \in \mathfrak{M}$, so does every scalar multiple M; 3) If M_1 and M_2 belong to $\mathfrak{M}$, so does their weakly closed absolutely convex cover $\overline{\Gamma(M_1, M_2)}$.

If $\mathfrak{M} = \delta$ is a family of all finite subsets of F, then the topology of a $\mathfrak{T}_{\mathfrak{M}}$-uniform convergence on sets of $\mathfrak{M}$ coincides with the weak topology $\sigma(E, F)$. If $\mathfrak{M}$ is the family of all $\sigma(F, E)$-bounded subsets of F, then the corresponding $\mathfrak{T}_{\mathfrak{M}}$ is the topology of the uniform convergence on sets from $\mathfrak{M}$ is called strong topology on E with respect to the duality of $\langle E, F\rangle$ and is denoted by $\beta(E, F)$. A locally convex topology $\mathfrak{T}$ on E is called compatible with duality $\langle E, F\rangle$, if dual to $(E, \mathfrak{T})$ coincides with F. The locally convex topology $\mathfrak{T}$ on E is compatible with duality $\langle E, F\rangle$ if and only if $\mathfrak{T}$ is the topology of uniform convergence for some saturated family $\mathfrak{M}$, covering F and formed $\sigma(F, E)$-relative compact subsets of F. Since weakly bounded subsets in F may not be relatively compact in the topology $\sigma(F, E)$, then the strong topology $\beta(E, F)$, is not compatible with duality $\langle E, F\rangle$. Further, there is the strongest locally convex topology on E compatible with $\langle E, F\rangle$, and namely, the topology of uniform convergence on all $\sigma(F, E)$-compact, absolutely convex subsets of F. This topology on E is called the Mackey topology with respect to $\langle E, F\rangle$ and is denoted $\tau(E, F)$. An LCS $(E, \mathfrak{T})$ is called a Mackey space if its topology is $\tau(E, E')$. A closed absorbent and absolutely convex subset in an LCS is called barrel. An LCS $(E, \mathfrak{T})$ is called barrelled if every barrel is neighborhood of zero. Fréchet spaces are barrelled spaces. There are however incomplete normed nonbarrelled spaces.

An LCS E is called bornological, if every absolutely convex subset absorbing all bounded sets in E is a neighborhood of zero. In other words, bornological space is such an LCS on which every seminorm bounded on bounded sets is continuous. Metrizable locally convex spaces are bornological. Note also that if E is an LCS that is barreled or Bornological, then E is a Mackey space. The topology of barrelled LCS coincides with the strong topology $\beta(E, E')$.

Let E be an LCS. Topologies $\beta(E, E')$ and $\beta(E,' E)$ are called strong topologies on E and E', respectively. The space $(E', \beta(E'E))$ is called the strong dual space to E. In [82], to denote strong topologies on E and E' the following notations are used: $\mathfrak{T}_b(E')$ and $\mathfrak{T}_b(E)$. Next, through E_σ and E_β also denote the space E with topologies $\sigma(E, E')$ and $\beta(E, E')$ respectively, and E'_σ and E'_β denote space E' with topologies $\sigma(E', E)$ and $\beta(E,' E)$, respectively. Through $\mathfrak{T}_c(E)$ denotes the topology on E' of uniform convergence on precompact sets from E. The topology on E of strong precompact convergence on precompact sets of space $(E', \beta(E'E))$ is denoted by $\mathfrak{T}_c(E')$. Dual to the space E'_β is called the second dual to an LCS E and is denoted by the symbol E''. On the second dual space E'', the topology can be introduced in various ways. If on E'' is defined the topology of uniform convergence on strongly bounded subsets of the space E', i.e. on bounded sets of the space E'_β, then E'' is called the strong second dual space to E. It is denoted by the symbol $(E'', \beta(E'',E'))$ or $(E'_\beta)'_\beta$. In [82] the strong second dual space is denoted by the symbol $E''[\mathfrak{T}_b(E', E'')]$. If on E'' the topology of uniform convergence is given to a family of all equicontinuous subsets of the space E', then this topology is called natural topology and is denoted by $\mathfrak{T}_n(E')$. An LCS E is barrelled space if and only if every weakly bounded set of its dual space is equicontinuous, i.e. when the equality $\mathfrak{T}_n(E') = \beta(E'', E')$ is true. An LCS E is called quasi-barrelled if every bounded set of its strong dual space is equicontinuous. An LCS is called semi-reflexive if $E = E''$. An LCS $(E, \mathfrak{T})$ is called reflexive if it is semi-reflexive and $\mathfrak{T} = \beta(E'', E')$. An LCS is called totally reflexive if it is reflexive together with its quotient space. An LCS is called a Montel or, in short, (M)-space, if it is separated and every bounded set in it is relatively compact. A Fréchet space of type (M) is called a Fréchet-Montel space or (FM)-space. Strong dual to space of type (M) is again a space of type (M). An LCS E is called distinguished if every bounded set B_1 of the space $(E'', \beta(E'', E'))$ is contained in bipolar of some bounded set $B \subset E$, i.e. $B_1 \subset B^{0E'0E''}$. It is well known ([82], p. 306) that the distinguishedness of an LCS is equivalent to the barrelledness of the strong dual space E'_β. An LCS $(E, \mathfrak{T})$ is called quasi-normable if for each equicontinuous set $M \subset E'$ there is a neighborhood V of zero in E such that the topology induced in M by the strong topology of the space E'_β coincides with the topology of uniform convergence on V. If E is quasi-barrelled, then this means that the strong dual E'_β satisfies the strict Mackey convergence condition. The LCS E satisfies the strict Mackey convergence condition, if for each bounded set $C \subset E$ there exists a closed absolutely convex bounded set $B \supset C$ such that the topology induced in C from E, coincides with the topology induced from the normed space E_B defined below.

Let E be an LCS and $B \neq 0$ be absolutely convex and bounded subset in E, then $E_B = \bigcup\limits_{n\in\mathbb{N}} nB$ is a (not necessarily closed) subspace of E. The Minkowski functional p_B of a set B in E_B is the norm. The normed space (E_B, p_B) will be further denoted by E_B. It is easy to see that the identity mapping $I_B : E_B \to E$ (called the canonical embedding) is continuous. Moreover, if B is complete in E, then E_B is a Banach space. Let now B and C are absolutely convex and bounded sets of LCS E such that $0 \neq B \subset C$. Then $E_B \subset E_C$ and the canonical embedding $I_{BC} : E_B \to E_C$ is continuous. Obviously that the relations $I_B = I_C \circ I_{BC}$ hold. Due to properties of bounded sets for the family $\{E_B;\ B \in \mathfrak{B}(E)\}$ pair $(\{E_B\}, \{I_{BC}\})$ forms an inductive family. According to ([82], Theorem 2, p. 381, and Theorem 11, p. 219), if the space $(E, \mathfrak{T})$ is bornological, then the space $(E, \mathfrak{T})$ can be represented as an inductive limit of inductive pair $(\{E_B\}, \{I_{BC}\});$ $B, C \in \mathfrak{B}(E))$. This fact is denoted as $(E, \mathfrak{T}) = \varinjlim I_{BC}(E_B)$.

Let $T : E \to G$ be a continuous map of the LCS E to the space G. Adjoint (transposed) mapping $T' : G' \to E'$ is defined using equality

$$\langle Tx, g' \rangle = \langle x, T'g' \rangle,$$

which is assumed to be valid for all $x \in E$ and $g' \in G'$.

Let us give examples of adjoint to some simple mappings. Let E be a Fréchet space, H its subspace, and $I : H \to E$ an embedding of H into E. Let R be a mapping that corresponds to each $x' \in E'$ its restriction $x'_0 \in H'$. From the relations

$$\langle x_0', y\rangle = \langle Rx', y\rangle = \langle x', Iy\rangle,$$

valid for all $y \in H$ and $x' \in E'$, it follows that R is the adjoint of I, i.e. $R = I'$.

Now let H be a closed subspace of E. If u' is a linear continuous functional on E/H, then by the equality $\langle u, x\rangle = \langle u', \widehat{x}\rangle$, $\widehat{x} \in E/H$, $x \in \widehat{x}$ the linear functional u on E is defined. If $u = Iu'$ is an embedding of $(E/H)'$ into E' and $K : E \to E/H$ is a canonical homomorphism, then the equalities

$$\langle u', \widehat{x}\rangle = \langle u', Kx\rangle = \langle Iu', x\rangle$$

show that I is the adjoint of K, i.e. $I = K'$. In particular, the adjoint of $K_n : E \to E/\operatorname{Ker} p_n$ is $K_n' = I_n : (E/\operatorname{Ker} p_n)' = \operatorname{Ker} p_n^{\perp} \to E'$. The restriction of K_n' to $(E_n, \widehat{p}_n)'$ is a mapping on $E'_{U_n^0}$, where $U_n = \{x \in E;\ p_n(x) \leq 1\}$. Let us show that $\pi'_{nm} : E_n' \to E_m'$ is an identical embedding. Indeed, let $u' \in E_n' = (E/\operatorname{Ker} p_n, \widehat{p}_n)'$, then the equality $\langle u', K_n x\rangle = \langle K_n' u', x\rangle$ defines on $E'_{U_n^0} \subset E'$ a linear continuous functional. From equalities

$$\langle u', K_n x\rangle = \langle u', \pi_{nm} K_m x\rangle = \langle \pi'_{nm} u', K_m x\rangle$$

it follows that $\pi'_{nm} u'$ is the identical image of the above-mentioned functional u' from E_n' to E_m'.

Section 2.7 also is devoted to the studies of the topological adjoint mapping of arbitrary homomorphisms.

Using adjoint mappings, we construct an important example of an inductive pair.

Let $(E, \mathfrak{T})$ be an LCS with basis of zero's neighborhoods $U_0(E) = \{U_\alpha;\ \alpha \in A\}$. For each $U_\alpha \in U_0(E)$ denote by $E'_{U_\alpha^0}$ Banach space with unit ball U_α^0, where U_α^0 are polars U_α to E'. Let again $p_\alpha(\cdot)$ be the Minkowski functional for U_α, $k_\alpha : E \to E/\operatorname{Ker} p_\alpha$ be canonical mapping and $E_\alpha = (E/\operatorname{Ker} p_\alpha . \widehat{p}_\alpha)$. It is well known ([82], p. 276) that $E'_{U_\alpha^0}$ is isomorphic to the Banach space E_α' and this isomorphism is realized by mapping k_α', which is conjugate to k_α and defined on E_α' by equality

$$\langle k_\alpha' x', y\rangle = \langle x', k_\alpha y\rangle \quad \text{for all} \quad x' \in E_\alpha' \quad \text{and} \quad y \in E\,.$$

Let $I_{U_\alpha^0 U_\beta^0} : E'_{U_\alpha^0} \to E'_{U_\beta^0}$ $(U_\alpha^0 \subset U_\beta^0)$ be identical embedding. Let us now show that the mapping $I_{U_\alpha^0 U_\beta^0}$ $(U_\alpha^0 \subset U_\beta^0)$ coincides with the mapping $\pi'_{\alpha\beta}$ which is conjugate to the extension of canonical mapping $\overline{\pi}_{\alpha\beta} : E_\beta \to E_\alpha$ $(\beta \geq \alpha)$. As noted above, every linear continuous functional $F \in E_\alpha' = (E/\operatorname{Ker} p_\alpha, \widehat{p}_\alpha)'$

by the equality $\langle F, k_\alpha x\rangle = \langle k'_\alpha F, x\rangle$ defines a linear continuous functional on E. From equalities

$$\langle k'_\alpha F, x\rangle = \langle F, k_\alpha x\rangle = \langle F, \overline{\pi}_{\alpha\beta} k_\beta x\rangle = \langle \pi'_{\alpha\beta} F, k_\beta x\rangle = \langle k'_\beta \overline{\pi}'_{\alpha\beta} F, x\rangle$$

it follows that $\overline{\pi}'_{\alpha\beta} F$ is the identical image of the mentioned functional in $E'_{u^0_\beta}$. From this we obtain that for the family $\{E'_{u^0_\alpha};\ \alpha \in A\}$, a pair

$$(\{E'_{u^0_\alpha}\}_{\alpha\in A}, \{I_{u^0_\alpha u^0_\beta}\}_{\alpha\le\beta}) \text{ or } (\{E'_\alpha\}_{\alpha\in A}, \{\overline{\pi}'_{\alpha\beta}\}_{\alpha\le\beta})$$

is inductive.

An LCS $(E, \mathfrak{T})$ is called strictly distinguished [199] if the space $E'_\beta = s \cdot \lim\limits_{\rightarrow} \overline{\pi}'_{\alpha\beta}(E'_\alpha)$ (sometimes identity mappings are omitted in the notation). It is obvious that a strictly distinguished LCS $(E, \mathfrak{T})$ is distinguished, since the space E'_β is barreled, as the inductive limit of the sequence of Banach spaces.

Let now $(E, \mathfrak{T})$ be a Fréchet space. The strong dual of a metrizable space is a (DF)-space. Moreover, according to A. Grothendieck [65], an LCS F is called a (DF)-space if F has fundamental sequence of bounded sets and each strongly bounded set that is a countable union of equicontinuous sets in F', is equicontinuous. The class of (DF)-spaces includes all normable spaces and all quasi-barreled spaces that have a fundamental sequence of bounded sets. Details (DF)-spaces are discussed in ([65], see also [147]). It is well known [65] that the distinguishedness of the Fréchet space is equivalent to the barrelledness or bornologicality of a strong dual space E'_β that is the inductive limit of a sequence of Banach spaces. We define a strictly regular Fréchet space as follows: the Fréchet space $(E, \mathfrak{T})$ is called strictly regular, if in E there is a basis of neighborhoods of zero $\{V_n\}$ such that $E'_\beta = s \cdot \lim\limits_{\rightarrow} I_{V^0_n V^0_m}(E'_{V^0_n})$ or $E'_\beta = s \cdot \lim\limits_{\rightarrow} E'_{V^0_n}$, i.e. the strong dual space E'_β of the Fréchet space E is the strict inductive limit of the canonical sequence of Banach spaces $\{E_{V^0_n}\}$. These Fréchet spaces occupy an important place in this book. In particular, the well-known in Banach spaces theorems of James and Bishop-Phelps allow natural extension for such spaces. On the other side, these theorems and their extensions are actually conditions for the existence of spline and spline algorithms in Fréchet space discussed in Chapter III (see Sections 1.3 and 3.2). It should also be noted that in the 80s of the 20th century these spaces became the subject of intensive research in connection with various tasks. Therefore, we begin the study of strict inductive limits and strictly regular Fréchet spaces, which were subsequently called *quojections*.

2.2 Definitions and some properties of strict projective and inductive limits

Projective topologies. Let us fix a linear space E, a family of separated locally convex spaces $\{(E_\alpha, \mathfrak{T}_\alpha);\ \alpha \in A\}$ and linear mappings $f_\alpha : E \to (E_\alpha, \mathfrak{T}_\alpha)$ such that for each $x \neq 0$ there is at least one index $\alpha \in A$, for which $f_\alpha(x) \neq 0$. Separated LCS $(E, \mathfrak{T})$ is called a locally convex kernel of the spaces $(E_\alpha, \mathfrak{T}_\alpha)$, and as $\mathfrak{T}$ is called the topology of the kernel with respect to the mappings f_α, if $\mathfrak{T}$ is the weakest locally convex topology on E, for which the mappings f_α are continuous. This fact is denoted as $(E, \mathfrak{T}) = \mathbb{K} f_\alpha^{-1}(E_\alpha, \mathfrak{T}_\alpha)$ ([82], p. 295). $\mathfrak{T}$-neighborhoods in E are f_α^{-1} inverse images of $\mathfrak{T}_\alpha$-neighborhoods of U_α and their finite intersections form a basis of $\mathfrak{T}$-neighborhoods. In ([147], p. 68) the topology $\mathfrak{T}$ is called a projective topology with respect to family $\{(E_\alpha, \mathfrak{T}_\alpha, f_\alpha);\ \alpha \in A\}$. It is obvious that the topology $\mathfrak{T}$ is the upper bound (in the lattice of topologies on E) for families of topologies $\{f_\alpha^{-1}(\mathfrak{T}_\alpha); \alpha \in A\}$.

Let us give the most important examples of projective topologies.

Products. Let $\{(E_\alpha, \mathfrak{T}_\alpha);\ \alpha \in A\}$ be a family of locally convex spaces. Topology of the product $\mathfrak{T}$ on space $E = \prod\limits_{\alpha \in A} E_\alpha$ is the projective topology on E relative to the projections $\pi_\alpha : E \to E_\alpha$.

Subspaces. Let G be a subspace of LCS $(E, \mathfrak{T})$. Topology induced by $\mathfrak{T}$ on G is a projective topology with respect to the canonical embedding $I : G \to E$. It is denoted by $\mathfrak{T} \cap G$ (in [82] it is denoted via $\widehat{\mathfrak{T}}$).

Let G be a subspace of the LCS E and $p(\cdot)$ be a continuous seminorm on E. We will denote by p_G restriction of the seminorm p on G. Let's say that G is a normable subspace with respect to the seminorm p if the set $\{x \in E;\ p(x) \le 1\} \cap G$ is a bounded neighborhood in the $\mathfrak{T} \cap G$ topology. If p_G is norm on G and (G, p_G) is complete, then G is called Banach subspace of the space E.

Weak topologies. Let E be a linear space and F be a non-empty subset of its algebraic dual space E^*, i.e. the set of all linear functionals on E. Kernel topology (projective topology) on E with respect to the family $\{(E_f, f);\ f \in F\}$, where E_f is the image of E under the mapping f, is called the weak topology generated by F, and is denoted by $\sigma(E, F)$. It is well known that E will be an LCS in topology $\sigma(E, F)$ if and only if F separates the points of E. In particular, if $(E, \mathfrak{T})$ is an LCS, then its dual space E' separates the points of E. The topology $\sigma(E, E')$ is called weak topology of the space E. On the other hand, if E is considered as a subspace of E'^*, then on E' we can define topology $\sigma(E', E)$ and the space $(E', \sigma(E', E))$ is LCS called the weak dual of $(E, \mathfrak{T})$.

Let A be a set of indices directed by a relation (reflexive, transitive, antisymmetric) "$\le$". Let $\{(E_\alpha, \mathfrak{T}_\alpha);\ \alpha \in A\}$ be a family of locally convex spaces, $\pi_{\alpha\beta}$ with $(\alpha \le \beta)$ denote a continuous linear map $(E_\beta, \mathfrak{T}_\beta)$ into $(E_\alpha, \mathfrak{T}_\alpha)$ such that

$\pi_{\alpha\alpha}$ is identity mapping and $\pi_{\alpha\beta} \cdot \pi_{\beta\gamma} = \pi_{\alpha\gamma}$ $(\alpha \leq \beta \leq \gamma)$. Then the pair $(\{E_\alpha\}_{\alpha\in A}, \{\pi_{\alpha\beta}\}_{\alpha\leq\beta})$ is called projective family.

Let E be the subspace of the product $\prod\limits_{\alpha\in A} E_\alpha$, whose elements $x = \{x_\alpha\}$ satisfy the relation $x_\alpha = \pi_{\alpha\beta} x_\beta$ for all $\alpha \leq \beta$. The space $(E, \mathfrak{T})$ is called the projective limit of projective family $(\{E_\alpha\}_{\alpha\in A}, \{\pi_{\alpha\beta}\}_{\alpha\leq\beta})$ with respect to the mappings $\pi_{\alpha\beta}(\alpha, \beta \in A,\ \alpha \leq \beta)$ and is denoted by $(E, \mathfrak{T}) = \varprojlim \pi_{\alpha\beta}(E_\beta)$ (sometimes the notation $(E, \mathfrak{T}) = \operatorname{proj}(E_\alpha, \mathfrak{T}_\alpha)$ is also used). From the above implies that the topology $\mathfrak{T}$ of the space E is projective topology with respect to the family $\{(E_\alpha,\ \mathfrak{T}_\alpha, \widehat{\pi}_\alpha);\ \alpha \in A\}$, where $\widehat{\pi}_\alpha$ is the restriction on E of the projection $\pi_\alpha : \prod\limits_{\beta\in A} E_\beta \to E_\alpha$. Projective limit $(E, \mathfrak{T})$ is called reduced if $\widehat{\pi}_\alpha(E)$ is everywhere dense in E_α for every $\alpha \in A$. The projective limit is called strict if the mappings $\widehat{\pi}_\alpha : E \to E_\alpha$ are surjective and open. Moreover, the linear mapping T of LCS E into LCS F is called open if for every open subset $U \subset E$ the image $T(U)$ is an open subset of $T(E)$ (in the topology induced F). Linear, continuous and open mapping $T : E \to F$ is called a topological homomorphism (or simply a homomorphism when this does not cause confusion). Examples of topological homomorphisms are the canonical (quotient) mapping $K : E \to E/M$, where M is a closed subspace of space E and the natural embedding $I : G \to E$, where G is a subspace of E. If a homomorphism is injective, then it is called a monomorphism (isomorphism in). Topological homomorphisms and their adjoint maps are studied intensively in Section 2.7.

Theorem 2.2.1 ([147], p. 71)**.** *Any complete LCS $(E, \mathfrak{T})$ is isomorphic to the projective limit of a family of Banach spaces.*

Indeed, let $\{U_\alpha;\ \alpha \in A\}$ be a generating family of absolutely convex neighborhoods of zero for the topology $\mathfrak{T}$, $p_\alpha(\cdot)$ be Minkowski functional for U_α, $\operatorname{Ker} p_\alpha = p_\alpha^{-1}(0)$, $K_\alpha : E \to E/\operatorname{Ker} p_\alpha$ be canonical mapping, $\widehat{p}_\alpha(K_\alpha x) = p_\alpha(x)$ be norm on quotient space $E/\operatorname{Ker} p_\alpha$ associated with the seminorm p_α, $E_\alpha = (\widetilde{E/\operatorname{Ker} p_\alpha, \widehat{p}_\alpha})$ is completion of the normed space $(E/\operatorname{Ker} p_\alpha, \widehat{p}_\alpha)$ and $\pi_{\alpha\beta} : (E/\operatorname{Ker} p_\beta, \widehat{p}_\beta) \to (E/\operatorname{Ker} p_\alpha, \widehat{p}_\alpha)$ is canonical mapping corresponding to each class of the equivalence $K_\beta(x)$ the class $K_\alpha(x)$ for $U_\beta \subset U_\alpha$ $(\alpha \leq \beta)$. This correspondence $K_\beta(x) \to K_\alpha(x)$ is continuous, since $\widehat{p}_\alpha(K_\alpha x) \leq \widehat{p}_\beta(K_\beta x)$ and therefore has a continuous continuation $\overline{\pi}_{\alpha\beta} : E_\beta \to E_\alpha$. In this case, the equalities are valid $\pi_{\alpha\beta} \cdot K_\beta = K_\alpha$ at $U_\beta \subset U_\alpha$ $(\alpha \leq \beta)$. This means that $(E, \mathfrak{T}) = \varprojlim \pi_{\alpha\beta}(E_\beta)$.

An LCS is called nuclear if for every α there is a $\beta \geq \alpha$ such that the mapping $\pi_{\alpha\beta}$ is nuclear, i.e. having the form

$$\pi_{\alpha\beta} K_\beta x = \sum c_j \langle K_\beta x, f_j' \rangle y_j,$$

where $\{f'_j\}$ is an equicontinuous sequence of continuous linear functionals of $E'_\beta = (E/\operatorname{Ker} p_\beta, \widehat{p}_\beta)$ and $\{c_j\}$ is a sequence of non-negative numbers such that $\sum\limits_{j=1}^{\infty} c_j < \infty$.

A Fréchet space is called *countable-normed* [63] if its topology is generated by a sequence of compatible norms. This means that the mappings $\widehat{\pi}_{nm}$ $(m \geq n)$ can be chosen to be injective. A Fréchet space is called *countably-Hilbert* if it is countable-normed and its norms are generated by inner products. An example of a nuclear Fréchet space with continuous norm that is not countably-Hilbert was constructed in [51].

Corollary. *Any Fréchet space $(E, \mathfrak{T})$ with a generating non-decreasing sequence of seminorms $\{p_n\}$ is isomorphic to the projective limit of a sequence of Banach spaces $E_n = (\widetilde{E/\mathrm{Ker} p_n}, \widehat{p}_n)$ with respect to mappings $\overline{\pi}_{nm} : E_m \to E_n$ $(n \leq m)$ and $K_n : E \to E_n$.*

We will use these notations for Fréchet spaces $(E, \mathfrak{T})$ without any further explanation. According to [15], a Fréchet space $(E, \mathfrak{T})$ is called a quojection if it is isomorphic to the projective limit of a sequence of Banach spaces with respect to surjective mappings. It means, that the mappings π_{nm} $(n \leq m)$ and K_n are homomorphisms. Therefore, the quojection is a strict projective limit of the sequence of Banach spaces $\{E_n\}$ and this fact is denoted via $(E, \mathfrak{T}) = s \cdot \varprojlim \pi_{nm}(E_m)$. Quojections are studied in the following Section 2.3.

An LCS $(E, \mathfrak{T})$ is called a *Schwartz space* or a *space of type* (S), if for each neighborhood U in E there is a neighborhood V completely bounded with respect to U, i.e. for every $\varepsilon > 0$ there is a finite set M_ε such that $V \subset M_\varepsilon + \varepsilon U$.

Fréchet spaces of type (S) are studied in sufficient detail in [65].

Let $\{E_\alpha;\ \alpha \in A\}$ be a directed system of subspaces of LCS E $(\alpha \leq \beta$, if $E_\alpha \subset E_\beta)$ and $E = \bigcup\limits_{\alpha \in A} E_\alpha$. Besides, on each E_α $(\alpha \in A)$, it is given a topology $\mathfrak{T}_\alpha$ such that for $\alpha \leq \beta$ the topology induced from $\mathfrak{T}_\beta$ on E_α is weaker than topology $\mathfrak{T}_\alpha$. Let us denote by I_α a canonical embedding of E_α into E and by $I_{\alpha\beta}$ a canonical embedding of E_α in E_β $(\alpha \leq \beta)$. Let $I_{\alpha\beta} \cdot I_{\beta\gamma} = I_{\alpha\gamma}$ for $\alpha \leq \beta \leq \gamma$. In this case, the couple $\{(E_\alpha, I_\alpha)_{\alpha \in A}; (I_{\alpha\beta})_{\alpha \leq \beta}\}$ is called inductive system. The space $(E, \mathfrak{T})$ is called the inductive limit of inductive system $\{(E_\alpha, I_\alpha)_{\alpha \in A}; (I_{\alpha\beta})_{\alpha \leq \beta}\}$ and $\mathfrak{T}$-topology of the inductive limit with respect to the system $I_{\alpha\beta}$, if $\mathfrak{T}$ is the finest locally convex topology for which the mappings I_α are continuous and $(E, \mathfrak{T})$ is Hausdorff space. This is denoted by $(E, \mathfrak{T}) = \varinjlim I_{\alpha\beta}(E_\alpha)$ or so $(E, \mathfrak{T}) = \varinjlim E_\alpha$ (The designation is also used $(E, \mathfrak{T}) = \underset{\to}{\operatorname{ind}} (E_\alpha, \mathfrak{T}_\alpha)$). Inductive system $\{(E, \mathfrak{T}_\alpha)_{\alpha \in A}, (I_{\alpha\beta})_{\alpha \leq \beta}\}$ is said to be strict if $\mathfrak{T}_\beta$ induces $\mathfrak{T}_\alpha$ on every E_α for $\alpha \le \beta$, i.e. the mappings $I_{\alpha\beta}$ $(\alpha \le \beta)$ are topological monomorphisms. Inductive limit is called strict if $(E, \mathfrak{T})$ is the limit of strict inductive system. This is denoted by $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow} I_{\alpha\beta}(E_\alpha)$ or $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow} E_\alpha$.

Inductive limit of increasing sequence of Fréchet spaces (respectively Banach spaces, respectively Hilbert spaces) is called a (LF)-space (respectively (LB)-space, respectively (LH)-space). Strict inductive limit of increasing strict inductive sequence of Fréchet spaces (respectively Banach spaces, respectively Hilbert spaces) is called a strict (LF)-space (strict (LB)-space, strict (LH)-space). Let us give examples of inductive topologies.

Quotient space. Let $(E, \mathfrak{T})$ be an LCS, G be a subspace and K be a canonical mapping from E on quotient space E/G, i.e. such a mapping which to everyone $x \in E$ puts in correspondence its equivalence class $\widehat{x} = x + G$. Quotient topology is defined as the strongest separated topology on E/G in which K is continuous. It is well known that this topology is separated if and only if G is closed. Hence, in the case of closedness of G the quotient topology is inductive topology with respect to the family $\{(E, \mathfrak{T}), K\}$. If $\{p_\alpha(\cdot), \alpha \in A\}$ forms a generating family of seminorms on E for topologies $\mathfrak{T}$, then the system of seminorms $\widehat{p}_\alpha(\widehat{x}) = \inf\{p_\alpha(x + g);\ g \in G\}$ generates quotient topology $\mathfrak{T}/G$ on E/G. In [82] quotient topology of topology $\mathfrak{T}$ is denoted by $\widehat{\mathfrak{T}}$.

Locally convex direct sums. Let $\{(E_\alpha, \mathfrak{T}_\alpha);\ \alpha \in A\}$ be family of linear spaces, then the algebraic direct sum $\oplus_{\alpha\in A} E_\alpha$ is defined as a subspace $\prod\limits_{\alpha\in A} E_\alpha$ whose elements $x = \{x_\alpha\}$ have no more than a finite number of nonzero projections $x_\alpha = \pi_\alpha(x)$. Let us denote by I_α the canonical embedding $E_\alpha \to \oplus_{\beta\in A} E_\beta$. Locally convex direct sum of family of locally convex spaces $\{(E_\alpha, \mathfrak{T}_\alpha);\ \alpha \in A\}$ is defined as $\oplus_{\alpha\in A} E_\alpha$ in inductive topology with respect to the family $\{(E_\alpha, \mathfrak{T}_\alpha, I_\alpha);\ \alpha \in A\}$ and is denoted by $(E, \mathfrak{T}) = \oplus_{\alpha\in A}(E_\alpha, \mathfrak{T}_\alpha)$.

Proposition 2.2.2. *Let $(E, \mathfrak{T})$ be a strict inductive limit of an increasing sequence of complete locally convex spaces $\{(E_n, \mathfrak{T}_n)\}$. Then the following statements are equivalent:*

a) *$(E, \mathfrak{T})$ is a strict (LB)-space, i.e. $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$, where $(E_n, \mathfrak{T}_n)$ are Banach spaces with the norm $\|\cdot\|_n$.*

b) *There is an absolutely convex $\mathfrak{T}$-neighborhood $U = \bigcup\limits_{n\in\mathbb{N}} U_n$ in $(E, \mathfrak{T})$, where $U_n = U \cap E_n$ is bounded absolutely convex neighborhood in $(E_n, \mathfrak{T}_n)$.*

c) *On $(E, \mathfrak{T})$ there is a continuous norm $\|\cdot\|$, inducing on each E_n the topology $\mathfrak{T}_n$ which is generated by the norm $\|\cdot\|_n$.*

Proof. a) $\Rightarrow$ b). According to the condition, there are bounded absolutely convex

$\mathfrak{T}_1$-neighborhood U_1 and $\mathfrak{T}_2$-neighborhood W such that $W \cap E_1 \subset U_1$. Consider the $\mathfrak{T}_2$-neighborhood $U_2 = \Gamma(U_1 \cup W)$, i.e. absolutely convex hull of the set $U_1 \cup W$ and prove that $U_2 \cap E_1 = U_1$. It is obvious that $U_1 \subset U_2 \cap E_1$. Let $x \in U_2 \cap E_1$, then $x = \alpha x_1 + \beta x_2$, where $x_1 \in U_1$, $x_2 \in W$ and $|\alpha| + |\beta| \leq 1$. From relation $\beta x_2 = x - \alpha x_1 \in E_1$ it follows that either $\beta = 0$ or $x_2 \in E_1$. In both cases we have that $x \in U_1$, i.e. $U_2 \cap E_1 \subset U_1$. By induction we can construct an increasing sequence of absolutely convex bounded $\mathfrak{T}_n$-neighborhoods U_n such that for each $n \in \mathbb{N}$ we will have $U_{n+1} \cap E_n = U_n$. It is obvious that then $\mathfrak{T}$-neighborhood $U = \bigcup\limits_{n\in\mathbb{N}} U_n$ for each $n \in \mathbb{N}$ satisfies the equality $U \cap E_n = U_n$.

b) $\Rightarrow$ a). By condition, in $(E, \mathfrak{T})$ there exists an absolutely convex neighborhood $U = \bigcup\limits_{n\in\mathbb{N}} U_n$, where $U_n = U \cap E_n$ is bounded absolutely convex neighborhood in LCS $(E_n, \mathfrak{T}_n)$. By virtue of the well-known theorem of A. N. Kolmogorov, locally convex spaces $(E_n, \mathfrak{T}_n)$ are normable and, by assumption, are Banach spaces. On the other hand, we have $U_n = U \cap E_n = U_{n+1} \cap E_n = (U \cap E_{n+1}) \cap E_n$, i.e. the topology of the space E_n coincides with induced topology from E_{n+1} to E_n. Hence, $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$.

The equivalence of $b) \Leftrightarrow c)$ is easily proved if we notice that the Minkowski functional $p_U(\cdot)$ of a neighborhood U is continuous norm on E satisfying condition of statement c) and vice versa. $\square$

Corollary 1. *Let $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \|\cdot\|_n)$ is a strict (LB)-space. In E there is a basis of $\mathfrak{T}$-neighborhoods of zero $U_0(E)$ such that for each neighborhood $U \in U_0(E)$ its Minkowski functional $\|\cdot\|_U$ is a norm on E inducing on every E_n norm topology $\|\cdot\|_n$. Normed space $(E, \|\cdot\|_U)$ is not complete; E_n are complete subspaces of $(E, \|\cdot\|_U)$ for each $n \in \mathbb{N}$ and $U \in U_0(E)$.*

The space $(E, \mathfrak{T})$ is a locally convex kernel of the family of normed spaces $\{(E, \|\cdot\|_U);\ U \in U_0(E)\}$ with respect to identity mappings $E \to (E, \|\cdot\|_U)$.

Corollary 2. *Let $(E, \mathfrak{T})$ be the inductive limit of increasing sequences of locally convex spaces $\{(E_n, \mathfrak{T}_n)\}$. Then the following statements are equivalent:*

a) *$(E, \mathfrak{T})$ is strict inductive limit sequence of normed spaces $\{(E_n, \|\cdot\|_n)\}$.*

b) *There is an absolutely convex $\mathfrak{T}$-neighborhood $U = \bigcup\limits_{n\in\mathbb{N}} U_n$ in $(E, \mathfrak{T})$, where $U_n = U \cap E_n$ is a bounded absolutely convex neighborhood in $(E_n, \mathfrak{T}_n)$.*

c) *On $(E, \mathfrak{T})$ there is a continuous norm $\|\cdot\|$ inducing on each E_n topology $\mathfrak{T}_n$.*

Proposition 2.2.3. *Let $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \|\cdot\|_n)$ be strict (LB)-space, F be an LCS and A be a linear map of E in F. Then the following statements are equivalent:*

a) *A is continuous.*

b) *A is sequentially continuous, i.e. from the condition $x_k \to x$ in E (and this means that $x_k, x \in E_{n_0}$ for all $k \in \mathbb{N}$ and some n_0 and $x_k \to x$ in E_{n_0}) it follows that $Ax_k \to Ax_0$ in F.*

c) *The restriction of A from $(E, \mathfrak{T})$ to E_n is continuous in the topology of norm $\|\cdot\|_n$ for each $n \in \mathbb{N}$, i.e. for each $n \in \mathbb{N}$ and each of a continuous norm q on F there is a constant $C_{n,q} > 0$ such that*

$$q(Ax) \leq C_{n,q}\|x\|_n \text{ for all } x \in E_n .$$

d) *Restriction of A from the normed space $(E, \|\cdot\|_U)$ on each E_n is continuous in the norm topology $\|\cdot\|_n$, where $U \in U_0(E)$ satisfies the conditions of Proposition 2.2.2.*

Proof. a)⇔b) is fair because $(E, \mathfrak{T})$ is bornological ([147], p. 82). a)⇔c) follows from Theorem 7 ([82], p. 398). a)⇔d) follows from Corollary 1 of Proposition 2.2.2, since the induced topologies from $(E, \mathfrak{T})$ and $(E, \|\cdot\|_U)$ on E_n coincide with the topology of the norm $\|\cdot\|_n$. □

2.3 Quojection is a strictly regular Fréchet space

This section studies Fréchet spaces representable in the form of a strict projective limit of the sequence Banach spaces and it is important when studying the problem of existence of generalized splines. As noted in 2.2, such spaces are called *quojections*.

Theorem 2.3.1. *Let $(E, \mathfrak{T})$ be the Fréchet space with the non-increasing generating sequence of closed absolutely convex $\mathfrak{T}$-neighborhoods $\{V_n\}$. Then the following statements are equivalent:*

a) *The space $(E, \mathfrak{T})$ is strictly regular, i.e. $(E', \beta(E', E)) = s \cdot \lim\limits_{\to} E'_{V_n^0}$.*

b) *In the space E there is a closed, bounded absolutely convex set B such that $B^0 \cap E'_{V_n^0}$ is bounded neighborhood in $E'_{V_n^0}$ for each $n \in \mathbb{N}$.*

c) *For every $n \in \mathbb{N}$, the $\mathfrak{T}$-neighborhoods has the form $V_n = \overline{B + \operatorname{Ker} p_n}$, where B is a closed, bounded and absolutely convex set and p_n is the Minkowski functional for V_n.*

Proof. a) ⇒ b). By condition we have $E'_\beta = s \cdot \lim\limits_{\to} E'_{V_n^0}$. By virtue of Proposition 2.2.2 we obtain the existence of a bounded set B, satisfying the conditions of statement b), if take into account the fact that strong neighborhoods are polars of bounded sets from E.

b) $\Rightarrow$ c). Let B be closed, bounded, absolutely convex set for which $B^0 \cap E'_{V_n^0} = B'_n$ is bounded neighborhood in $E'_{V_n^0}$. It is easy to check that homothetic images of the sets B'_n form a fundamental sequence of bounded sets in E'_β. Moving on to the polars in E, we obtain that ${B'_n}^0 = \overline{B^{00} + {E'_{V_n^0}}^\perp}$, where ${E'_{V_n^0}}^\perp$ is weak closed subspace in E, orthogonal to $E'_{V_n^0}$. Since B is closed and absolutely convex, then $B^{00} = B$ and therefore ${B'_n}^0 = \overline{B + {E'_{V_n^0}}^\perp}$. Let $p_n(\cdot)$ be the Minkowski functional for ${B'_n}^0$. Obviously, ${E'}^\perp_{V_n^0} \subset \operatorname{Ker} p_n$. If there exists $x \in \operatorname{Ker} p_n$ and $x \overline{\in} {E'_{V_n^0}}^\perp$, then $\lambda\, x \in {B'_n}^0$ for any number λ and there exists $x' \in E'_{V_n^0}$ such that $\langle x, x' \rangle \neq 0$. On the other hand, for some $\lambda_0 > 0$ we have that $\lambda_0 x' \in B'_n$ and therefore $\langle \lambda_0 x', x \rangle = \langle x', \lambda_0 x \rangle = 0$, which contradicts our assumption. Hence, ${E'_{V_n^0}}^\perp = \operatorname{Ker} p_n$. Now let us assume that $V_n = {B'_n}^0$, i.e. $V_n = \overline{B + \operatorname{Ker} p_n}$ for each $n \in \mathbb{N}$.

c) $\Rightarrow$ a). Let the condition of statement c) be satisfied, then $E'_{V_n^0} = \operatorname{Ker} p_n^\perp$. Indeed, passing to the polar in E', we obtain that $V_n^0 = B \cap \operatorname{Ker} p_n^\perp$. Therefore, any continuous linear functional, orthogonal to $\operatorname{Ker} p_n$ belongs to $E'_{V_n^0}$ and vice versa. On the other hand, in the (DF)-space E'_β bounded sets coincide with equicontinuous sets V_n^0. From the above equality $V_n^0 = B^0 \cap \operatorname{Ker} p_n^\perp$ it follows the metrizability of these sets, and by virtue of [65] we get that $E'_\beta = \varinjlim \operatorname{Ker} p_n^\perp$. From the equalities $V_n^0 = B^0 \cap \operatorname{Ker} p_n^\perp = (B^0 \cap \operatorname{Ker} p_{n+1}^\perp) \cap \operatorname{Ker} p_n^\perp$ it also follows that the identity mappings $\operatorname{Ker} p_n^\perp \to \operatorname{Ker} p_{n+1}^\perp$ are monomorphisms. This means that $E'_\beta = s \cdot \varinjlim \operatorname{Ker} p_n^\perp$, i.e. the space $(E, \mathfrak{T})$ is strictly regular. Therefore, strong dual of E'_β to a strictly regular Fréchet space is strict inductive limit of weakly closed sequences of Banach subspaces $\{\operatorname{Ker} p_n^\perp\}$ in E'_β. $\square$

It should be noted that in the case of reflexivity of the Fréchet space $(E, \mathfrak{T})$ in statements b) and c) can be assumed that B is weakly compact bounded set and, therefore, the sets $B + \operatorname{Ker} p_n$ will be closed in $(E, \mathfrak{T})$.

Corollary 1. *Let $(E, \mathfrak{T})$ be a strictly regular Fréchet space. Then the spaces $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ are Banach spaces, $K_n : E \to (E/\operatorname{Ker} p_n, \widehat{p}_n)$ is a homomorphism and its adjoint map K'_n is strong monomorphism.*

Indeed, by virtue of Theorem 2.3.1 the topology of the space $(E, \mathfrak{T})$ is generated by a sequence of neighborhoods of the form $V_n = \overline{B + \operatorname{Ker} p_n}$, where B is a closed, absolutely convex subset of the space E. Therefore, we have

$$\widehat{V}_{nn} = k_n(V_n) = k_n(\overline{B + \operatorname{Ker} p_n}) \subset \overline{k_n(B)}.$$

This means that $\widehat{V}_{nn}$ is a bounded, absolutely convex neighborhood in quotient space $E/\operatorname{Ker} p_n$, since it is contained in the closure of the image of a bounded set under a canonical mapping. It follows that quotient spaces $E/\operatorname{Ker} p_n$ are Banach in the norm $\widehat{p}_n$, i.e. $E_n = (E/\operatorname{Ker} p_n, \widehat{p}_n)$ are Banach spaces for every $n \in \mathbb{N}$.

The adjoint mapping k'_n is the identity embedding of space E'_n in E' with weakly closed image $k'_n(E'_n) = \operatorname{Ker} p_n^{\perp}$. It is well known that it is a weak monomorphism. From equalities $V_n^0 = \operatorname{Ker} p_n^{\perp} \cap B^0$ it follows that $k'{}_n$ is also strong monomorphism, i.e. strong topology $\beta(\operatorname{Ker} p_n^{\perp}, E_n)$ on $\operatorname{Ker} p_n^{\perp}$ of the dual system $\langle \operatorname{Ker} p_n^{\perp}, E_n\rangle$ coincides with induced topology $\beta(E', E) \cap \operatorname{Ker} p_n^{\perp}$ on $\operatorname{Ker} p_n^{\perp}$.

Corollary 2. *The class of strictly regular Fréchet spaces coincides with quojection.*

Indeed, if the Fréchet space $(E, \mathfrak{T})$ is strictly regular, then $E'_\beta = s\cdot\varinjlim \pi'_{nm}(E'_m)$ and mappings π'_{nm} $(n \leq m)$ are strong monomorphisms. By virtue of the well-known theorem of J. Dieudonne and L. Schwartz [43], we find that the mappings $\pi_{nm} : E_m \to E_n \;\; (n \leq m)$ are also surjective and therefore homomorphisms. Hence, the space $(E, \mathfrak{T})$ is a quojection.

Let $(E, \mathfrak{T})$ be a quojection, then the mappings $\pi_{nm} : E_m \to E_n \;\; (n \leq m)$ and $k_n : E \to E_n \;\; (n \in m)$ are homomorphisms and conjugate mappings $\pi'_{nm} : E'_n \to E'_m \;\; (n \leq m)$ are strong monomorphisms. To prove the strict regularity of the space $(E, \mathfrak{T})$ we present the reasoning due to the first author of the works [39]. More precisely, it will be proved that in $(E, \mathfrak{T})$ there exists generating sequence of absolutely convex neighborhoods, having the form $B + \operatorname{Ker} k_n$, where B is an absolutely convex bounded set. Indeed, since $\pi_{n,n+1}$ is surjective, then for each $n \in \mathbb{N}$, by induction we can find a sequence of bounded absolutely convex neighborhoods $B_n \in \mathfrak{B}(E_n) \cap U_0(E_n)$ such that $\pi_{n,n+1}(B_{n+1}) = B_n$ for all $n \in \mathbb{N}$. Let $B = \prod\limits_{n\in\mathbb{N}} B_n \cap E$. Obviously, B is bounded set in E. Let us prove that $U_n = k_n^{-1}(k_n(B_n)),\ (n \in \mathbb{N})$ forms a generating sequence of neighborhoods in E having the form $U_n = B + \operatorname{Ker} k_n$.

Let $n \in \mathbb{N}$. Then $k_n(B + \operatorname{Ker} k_n) \subset B_n$ because $\operatorname{Ker} k_n + B \subset U_n$. Conversely, let $x = \{x_k\}_{k\in\mathbb{N}} \in U_n$. Then $x \in E$ and $x_n \in B_n$. Since $\pi_{l,l+1}(B_{l+1}) = B_l$ we can by induction construct a sequence $\{y_l\}_{l\geq n} \in \prod\limits_{l\geq n} B_l$ such that $y_n = x_n$ and $\pi_{l,l+1}(y_{l+1}) = y_l \;\; (l \geq n)$. For all $l < n$ let's put $y_l = \pi_{ln}(x_n) \in \pi_{ln}(B_n) = B_l$. Then $y = \{y_l\}$ is contained in B and $k_n(x) = x_n = y_n = k_n(y)$, where $x = x - y + y \in \operatorname{Ker} k_n + B$. Obviously, $\{\frac{1}{n} U_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathfrak{T}$-neighborhoods of zero in E. Now, repeating the reasoning given when proving the implications c) $\Rightarrow$ a) of Theorem 2.3.1, we obtain a strict regularity of space E.

From Corollary 2 of Theorem 2.3.1 it follows that the class of strictly regular Fréchet spaces coincides with the class of Fréchet spaces representable in the form of a strict projective limit of a sequence of Banach spaces. In the work of V. Slovikovsky and V. Zavadovsky [149] strict projective limits were called as relatively complete B_0-spaces. These spaces have been intensively studied since the early 80s in the study of a wide variety of questions of functional analysis, and the term “quojection” became very widespread (see, e.g., [17, 20–25, 27, 28, 101–103, 110, 111, 117, 153, 154]). That’s why, we are forced in the future, instead of the previously proposed term “ strictly regular Fréchet space” use the term “quojection”. It should also be noted that in the works of many foreign authors [17, 20, 25, 26, 36, 38, 111] the identity of these notion was also noticed. It’s obvious that quojection (strictly regular Fréchet space) is distinguished. On the other hand, there are reflexive, and therefore distinguished Fréchet spaces, which are not quojections. These classes include:

1) Reflexive Fréchet spaces in which do not exist total bounded sets, since in their dual spaces do not exist continuous norms. An example of such a Fréchet space was constructed in [6].

2) Reflexive Fréchet spaces on which there exist continuous norms, since, obviously, continuous norms cannot exist on a quojection.

3) The Fréchet-Montel space (of type (FM)), is a quojection if and only if it is isomorphic to the nuclear space Fréchet of all sequences $\omega = R^N$ (or C^N). Indeed, let the strong conjugate E'_β to a space of type (FM) is a strict inductive limit of Banach spaces. Then the space E'_β is also Montel space and, by virtue of J. Dieudonne’s theorem ([82], p. 371), has fundamental sequence absolutely convex compact sets $\{V_n^0\}$. Therefore, the unit balls V_n^0 of spaces $E'_{V_n^0} = \operatorname{Ker} p_n^\perp$, are compact $Kerp_n^\perp$ is and finite-dimensional. Consequently, E'_β turns out to be a strict inductive limit of sequences of finite-dimensional Banach spaces $\operatorname{Ker} p_n^\perp$ and therefore isomorphic to the space of all finite sequences φ ([82], p.405) with the topology of the strict inductive limit. Then the strong second conjugate $(E'_\beta)'_\beta = E$ and is isomorphic to the space ω, which is conjugate to the space φ.

From the representation of neighborhoods of the quojection it follows that the quotient space of a quojection over any closed subspace is always a quojection. Since every Fréchet space is isomorphic to a closed subspace the product of a sequence of Banach spaces, which is quejection, then the closed subspace of a quejection is not always a quejection.

By virtue of ([65], Theorem 5), a sufficient condition for the distinguishedness of the Fréchet space $(E, \mathfrak{T})$ is metrizability of all bounded sets of the space E'_β or existence of an everywhere dense sequence in it (in space E'_β) ([65], Corollary 2 of Theorem 4). As shown by examples of spaces of type (FM), non-isomorphic ω, these conditions are no longer sufficient for $(E, \mathfrak{T})$ to be a quoejection, because in its strong dual space all bounded sets are metrizable [121], and there exists an everywhere dense sequence ([82], p. 371).

The quojection $(E, \mathfrak{T})$ is quasi-normable, however, the converse is false, i.e. there is a quasi-normable space that is not quojection. Indeed, the quasinormability of a quasi-barrel LCS $(E, \mathfrak{T})$ is equivalent to the fact that the space E'_β satisfies the strict Mackey convergence condition. It means, that for every equicontinuous set $C \subset E'_\beta$ there exists a $\mathfrak{T}$-neighborhood of zero V such that the topology induced in C from strong topology coincides with the topology of uniform convergence on V, i.e. with the topology induced from the normed space E'_{V^0}. As follows from Theorem 2.3.1, for quojection a stronger statement holds: for every equicontinuous set $C \subset E'_\beta$, there is $\mathfrak{T}$-neighborhood V such that $C \subset V^0$ and induced topology in E'_{V^o} with the strong topology $\beta(E', E)$ coincides with the topology of Banach space E'_{V^0}. An example of a quasi-normable space, which is not a quojection, serves any quasi-normable Fréchet-Montel space, not isomorphic to the space ω. In particular, such is Schwartz space $S(R^l)$ (see Section 2.6.2).

Theorem 2.3.2. *Let* $(F, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow} F_n$ *be strict* (LB)*-space. Then the following statements are valid:*

a) *A strong dual space* F'_β *is a quojection and* $F'_\beta = s \cdot \lim\limits_{\leftarrow} F'/F_n^\perp$.

b) *For the strong second dual space the following equalities hold:*

$$(F'', \beta(F'', F')) = s \cdot \lim_{\rightarrow} F''_n = s \cdot \lim_{\rightarrow} (F'/F_n^\perp)'.$$

Proof. a) Let V represent $\mathfrak{T}$-neighborhood satisfying condition b) of Proposition 2.2.2 and $V \cap F_n = V_n$. It is well known that homothetic images of the sets V_n form a fundamental sequence of bounded sets in $(F, \mathfrak{T})$. Passing to the polars in F' we obtain that $V_n = \overline{V^0 + F_n^\perp}$. Since V^0 is weakly compact and $F_n^\perp$ is weakly closed, then we immediately obtain weak closedness of set $V^0 + F_n^\perp$, i.e. $V_n^0 = V^0 + F_n^\perp$. The sequence $\{V_n^0\}$ generates the topology $\mathfrak{T}_\beta$, and V_n^0 are closed in this topology. Let p_n be the Minkovski functional for V_n^0. Similar to how it was done when proving the implications b) $\Rightarrow$ c) of Theorem 2.3.1 we obtain that $\operatorname{Ker} p_n = F_n^\perp$. Therefore, $V_n^0 = V^0 + \operatorname{Ker} p_n$, F'_β is quojection and $F'_\beta = s \cdot \lim\limits_{\leftarrow} F'/F_n^\perp$.

b) Due to the properties of quojection, this is a rare example of a projective limit of a sequence of Banach spaces, the strong dual of which is represented as a strict inductive limit of sequence of dual spaces. Therefore, the equalities are true

$$(F'', \beta(F'', F')) = s \cdot \lim_{\rightarrow} (F'/F_n^\perp)'_\beta = s \cdot \lim_{\rightarrow} (F'/\operatorname{Ker} p_n)'_\beta.$$

Further, every bounded set of quotient space $F'/\operatorname{Ker} p_n$ is contained in the canonical image of the bounded set V^0 from E', and this is equivalent to the fact that the space $(F'/\operatorname{Ker} p_n)'$ which is endowed with strong topology

$\beta((F'/\operatorname{Ker} p_n)', F'/\operatorname{Ker} p_n)$, is isomorphic to the subspace $\operatorname{Ker} p_n^{\perp} = F_n^{\perp\perp}$ of space E'', endowed with induced topology $\beta(F'', F') \cap F_n^{\perp\perp}$, in which it is a Banach space. On the other hand, F_n' was identified with the quotient space $F'/F_n^{\perp}$, endowed with quotient topology $\beta(F', F)/F_n^{\perp}$ due to known properties dual to subspaces of (DF)-spaces. Therefore, $F_n'' = F_n^{\perp\perp}$ and $(F'', \beta(F'', F')) = s \cdot \lim\limits_{\rightarrow} F''{}_n$. □

Corollary. *Let $(E, \mathfrak{T})$ be a Fréchet space, strongly dual space E'_β of which is strict (LB)-space. Then its strong bidual space $(E'', \beta(E'', E'))$ is a quojection.*

Statement a) of Theorem 2.3.2, which, by virtue of Corollary 2 of Theorem 2.2.2 is valid for the strict inductive limit of sequence of normed spaces, strengthens the result of A. Grothendieck ([65], corollary of Proposition 8), which proves the distinguishedness of the strong dual to the strict inductive limit of the sequence of normed spaces.

In [14], an example of a non-reflexive Fréchet space was constructed, which is not a quojection, but its strong dual space is a strict (LB)-space. Other examples and construction method of such Fréchet spaces can be found in [102]. Fréchet space, the strong bidual of which is a quojection, was named in [102] prequojection. In [199] (see also [38, 117]), it is proved that the class of prequojections exactly coincides with the class of Fréchet spaces, strong dual of which are strict (LB)-spaces.

Theorem 2.3.3. *Let $(F, \mathfrak{T})$ be a strict (LB)-space. In order that the quotient space F/G over a closed subspace G to be a strict (LB)-space in the quotient topology, it is necessary, and in the case of reflexivity of the quotient space F/G it is sufficient, that for a weakly closed subspace $G^{\perp}$ of quojection F'_β was a quojection.*

Proof. *Necessity*. Let quotient space F/G be a strict (LB)-space. By virtue of statement a) of Theorem 2.3.2, the strong dual space F'_β is a quojection. From ([65], Proposition 5) it follows that the subspace $(G^{\perp}, \beta(F', F) \cap G^{\perp})$ of the space F'_β is identified with the dual to the quotient space F/G in the strong topology $\beta(G^{\perp}, F/G)$, in which it is a quojection.

Sufficiency. Now, let the weakly closed subspace $G^{\perp}$ of the quojection F'_β be again a quojection. It is well known that for an arbitrary closed subspace G the quotient space $(F/G, \mathfrak{T}/G)$ is a (LB)-space. Further, as noted above, the strong topology $\beta(F', F) \cap G^{\perp}$ on $G^{\perp}$ coincides with the induced topology $\beta(F', F) \cap G^{\perp}$. Let in this topology $G^{\perp}$ be a Banach space, then, by virtue of Theorems 4 and 7 from ([82], p. 397), we find that F/G is also a Banach space. If $G^{\perp}$ is a non-normed subspace in F'_β, then, by assumption, it is a reflexive quojection, i.e.

strong bidual to the (LB)-space F/G is a strict (LB)-space and, therefore, it itself is a strict (LB)-space. □

Remark 2.3.1 (A note about the terms “strictly distinguished”, “strictly regular”, “quoection”, “prequojection”)**.** The term “strictly distinguished Fréchet space” was first introduced in the Russian paper [193] as a Fréchet space whose strong dual is strict (LB-space) that is a natural reinforcement of A. Grothendieck’s term “distinguished Fréchet space” ([82], pp. 306, 399). This article was translated from Russian to English and its translator correctly used the term “strictly distinguished”. Then the author used the term “strictly distinguished” in the paper [196], the translator of which used the term “strictly regular” to denote those Fréchet spaces whose strong dual is strict inductive limit of the sequence spanned on the polars of neighborhoods of zero (canonical sequence). Unfortunately, we did not see the translation correction. We used the same term for such spaces in [39]. Later, the term “quojection” was used to denote “strictly regular Fréchet spaces”. However, in [14], it was proved that the Fréchet spaces exist whose strong dual is strict (LB)-spaces, but they are not strict (LB)-spaces of canonical sequence.

Thus, the strictly regular and strictly distinguished Fréchet spaces are different from each other, and the Fréchet spaces whose strong dual are strict (LB)-spaces are called “prequojection”. At the same time, we introduced the notion of a strictly distinguished LCS as a space whose strong dual is strict inductive limit of the family of Banach spaces. In particular, in the case of Fréchet spaces, the prequojection and strictly distinguished Fréchet spaces are identical [199]. Unfortunately, we have not seen the English translation of this article.

2.3.1 Examples of quojections

A simple example of a quojection is the product of a sequence of Banach spaces $(E,\mathfrak{T}) = \prod\limits_{k\in\mathbb{N}} (E_k, \|\cdot\|_k)$. In particular, the simplest quojection is the space of all real (complex) sequences $\omega = R^N(C^N)$.

Indeed, let us define the topology of the space E with sequence of seminorms

$$p_n(x) = \sum_{k=1}^{n} \|x_k\|_k, \quad x = \{x_k\} \in E, \quad n \in \mathbb{N}.$$

It is easy to verify that the quotient space $E/\operatorname{Ker} p_n$ by quotient norm $\widehat{p}_n$ is isomorphic to the Banach space $\prod\limits_{k=1}^{n} (E_k, \|\cdot\|_k)$, i.e. $(E,\mathfrak{T})$ is a quojection. Strong dual space $E'_\beta = \oplus_{k\in\mathbb{N}}(E'_k, \|\cdot\|'_k)$, i.e. is the direct sum of strong dual Banach spaces.

According to [110], a quojection is called trivial if it isomorphic to the product of a sequence of Banach spaces. An example of a non-rivial quojection was constructed in [110].

It is easy to verify that the Fréchet space $(E, \mathfrak{T})$ is isomorphic to the space ω if and only if there exists on E sequence of seminorms generating a topology $\mathfrak{T}$ such that $\dim(E/\operatorname{Ker} p_n) = \operatorname{Co}\dim \operatorname{Ker} p_n < \infty$ for each $n \in \mathbb{N}$, i.e. space ω is represented as the projective limit of finite-dimensional spaces. The strong dual to the space ω is isomorphic to the space of all finite sequences φ.

Let us now consider the Fréchet space $B \times \omega$, where $(B, \|\cdot\|)$ is a Banach space. Let us define a sequence of seminorms

$$p_n(x^{(1)}, x^{(2)}) = (\|x^{(1)}\|^2 + |x_1^{(2)}|^2 + \cdots + |x_{n-1}^{(2)}|^2)^{1/2}),$$
$$x = (x^{(1)}, x^{(2)}) \in B \times \omega, \;\; n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ the quotient space $(B \times \omega)/\operatorname{Ker} p_n$ is a Banach space with the norm $\widehat{p}_n$ isomorphic to the space $(B \times R^{n-1}, \, p_{n,B\times R^{n-1}})$, where $p_{n,B\times R^{n-1}}$ is restriction of seminorms p_n on the subspace $B \times R^{n-1}$. Canonical mappings k_n and π_{nm} are topological homomorphisms, $\dim \operatorname{Ker} \pi_{nm} = m - n < \infty$ and therefore $\operatorname{Ker} \pi_{nm}$ have topological complement in E_m. The strong dual space E' isomorphic to the space $B' \times \varphi$. The space $B' \times \varphi$ in the strong topology admits the following representation $(B' \times \varphi)_\beta = s \cdot \lim\limits_{\to}(\operatorname{Ker} p_n^\perp, \|\cdot\|_n')$, i.e. $B' \times \varphi$ is represented as a strict inductive limit of a sequence of Banach spaces $\{(\operatorname{Ker} p_n^\perp, \|\cdot\|_n)\}$, where $\operatorname{Ker} p_n^\perp$ is considered in induced topology $\beta(B' \times \varphi, B \times \omega) \cap \operatorname{Ker} p_n^\perp$, which coincides with the topology of the norm $\|\cdot\|_n'$, generated by the polar $V_n^0 = \{x' \in B' \times \varphi; \;\; \sup\{|\langle v, x'\rangle|; \;\; v \in V_n\} \leq 1\}$. Adjoint mappings $\pi_{nm}' : (\operatorname{Ker} p_n^\perp, \|\cdot\|_n') \to (\operatorname{Ker} p_m^\perp, \|\cdot\|_m')$, and $k_n' : (\operatorname{Ker} p_n^\perp, \|\cdot\|_n') \to B' \times \varphi$ are topological monomorphisms. From the mentioned properties of mappings π_{nm} for π_{nm}' we obtain that $\dim \operatorname{Co}\ker \pi_{nm}' = m - n < \infty$, i.e. $\dim(\operatorname{Ker} p_m^\perp / \pi_{nm}'(\operatorname{Ker} p_n^\perp)) = \dim(\operatorname{Ker} p_m^\perp / \operatorname{Ker} p_n^\perp) = m - n < \infty$.

It is well known that every closed subspace G of the space $B \times \omega$ is again a space of the same type. Moreover, every closed subspace G of the Fréchet space E is a quojection if and only if it is isomorphic to the space $B \times \omega$. Indeed, if E is not isomorphic to the space $B \times \omega$, then due to [16] it has a nuclear Kothe subspace, i.e. nuclear Fréchet space with continuous norm and basis, which is not quojection.

The quotient space of the space $B \times \omega$ is again the space of the same form, since every closed subspace of the quotient space is isomorphic to the quotient space of the subspace.

Let us now present a description of the topology of the strong dual to subspace G of the space $B \times \omega$, which we will need in further. Let $p_{n,G}$ be the restriction of the seminorm p_n to G and $(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})$ is a normed space, where $\widehat{p}_{n,G}$ is associated norm on the quotient space. The space $(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})$ is isometric to the subspace $k_n(G)$ of the quotient space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ and this isometry is carried out by the correspondence $k_{n,G}(g) \to k_n(g)$, where $k_{n,G}$ is the canonical mapping of G to $G/\operatorname{Ker} p_{n,G}$. As is known, the subspace G is isomorphic to the projective limit of a sequence of Banach spaces $G_m = (k_m(G), \widehat{p}_{m,k_m(G)})$ with respect to the mappings $\overline{\pi}_{nm,G_m}$, where π_{nm,G_m} is a restriction of π_{nm} to G_m, $\overline{\pi}_{nm,G_m}$ is its continuous extension to $\widetilde{G}_m$, and $\widehat{p}_{m,k_m(G)}$ is restriction of $\widehat{p}_m$ to $k_m(G)$. Let us prove now that the conjugate mapping $\pi'_{nm,G_m} : G'_n \to G'_m$ is a monomorphism. Indeed, we have that $k_n(G)' = (E/\operatorname{Ker} p_n)'/k_n(G)^\perp = \operatorname{Ker} p_n^\perp/(\operatorname{Ker} p_n^\perp \cap G^\perp)$, since $k_n(G)^\perp = {k'_n}^{(-1)}(G^\perp) = \operatorname{Ker} p_n^\perp \cap G^\perp$. Therefore

$$\pi'_{nm,G_m} : (\operatorname{Ker} p_n^\perp, \|\cdot\|_n)/(\operatorname{Ker} p_n^\perp \cap G^\perp) \to (\operatorname{Ker} p_m^\perp, \|\cdot\|_m)/(\operatorname{Ker} p_m^\perp \cap G^\perp).$$

Applying Theorems 6 and 7 from ([82], p. 55), we obtain the algebraic isomorphism of the following quotient spaces

$$(\operatorname{Ker} p_m^\perp/(\operatorname{Ker} p_m^\perp \cap G^\perp))/(\operatorname{Ker} p_n^\perp/(\operatorname{Ker} p_n^\perp \cap G^\perp)),$$

$$(\operatorname{Ker} p_m^\perp/(\operatorname{Ker} p_m^\perp \cap G^\perp))/((\operatorname{Ker} p_n^\perp + G^\perp \cap \operatorname{Ker} p_m^\perp)/(\operatorname{Ker} p_m^\perp \cap G_m)),$$

and

$$\operatorname{Ker} p_m^\perp/(\operatorname{Ker} p_n^\perp + G^\perp \cap \operatorname{Ker} p_m^\perp).$$

But the last quotient space is the image of a finite-dimensional space $\operatorname{Ker} p_m^\perp/(\operatorname{Ker} p_n^\perp)$ under homomorphism and therefore itself is finite-dimensional, i.e.

$$\dim \operatorname{coker} \pi'_{nm,G_m} \leq m - n.$$

Besides, according to the theorem proved in [81], we obtain that the mapping π'_{nm,G_m} is a monomorphism and $\pi'_{nm,G_m}(\operatorname{Ker} p_n^\perp/(\operatorname{Ker} p_n^\perp \cap G^\perp))$ is closed in $\operatorname{Ker} p_m^\perp/(\operatorname{Ker} p_m^\perp \cap G^\perp)$. This means that $\operatorname{Ker} p_1^\perp/(\operatorname{Ker} p_1^\perp \cap G^\perp)$ has at most countable dimension in G'. Further, as proved in the note after Proposition 11 in [65], it turns out that G'_β is isomorphic to $B'_1 \times \varphi$, i.e. G is isomorphic to $B_1 \times \omega$.

In the case of reflexivity of space, we can prove the following assertion: the reflexive (LB)-space $(F, \mathfrak{T})$ has a quotient space that is not a strict (LB)-space if and only if it is not isomorphic to the space $B \times \varphi$, where B is reflexive Banach space. Indeed, if $(F, \mathfrak{T})$ has quotientspace $(F/G, \mathfrak{T}/G)$, which is reflexive (LB)-space, but is not a strict (LB)-space, then dual space $G^\perp$ in the strong topology $\beta(G^\perp, F/G)$ is identified with the space $(G^\perp, \beta(F', F) \cap G^\perp)$. Therefore, $G^\perp$

cannot be a quojection and therefore F'_β is not isomorphic to the space $B' \times \omega$, i.e. $(F, \mathfrak{T})$ is not isomorphic to the space $B \times \varphi$.

Let us also prove that if $(F, \mathfrak{T}) = B \times \omega$, then every closed subspace $(G, \beta(F', F) \cap G)$ of its strong dual subspace $B' \times \varphi$ is a strict (LB)-space.

Closed subspace $(G, \beta(F', F) \cap G)$ of space $B \times \varphi$ is a Banach subspace if and only if it is contained in some Banach space $(\operatorname{Ker} p_n^\perp, \|\cdot\|_n)$ and is a closed subspace in it. If $(G, \beta(F', F) \cap G)$ is a non-normed closed subspace, then $G = \bigcup\limits_{n \in \mathbb{N}} (G \cap \operatorname{Ker} p_n^\perp)$ and it should be proven that $(G, \beta(F', F) \cap G) = s \cdot \lim\limits_{\to} (G \cap \operatorname{Ker} p_n^\perp, \|\cdot\|_n)$. From the properties of monomorphisms π'_{nm} $(n \leq m)$ it follows that for monomorphisms $I_{nm} : (G \cap \operatorname{Ker} p_n^\perp, \|\cdot\|_n) \to (G \cap \operatorname{Ker} p_m^\perp, \|\cdot\|_m)$ the inequalities $\dim \operatorname{Co} ker I_{nm} \leq m - n < \infty$ are valid. Therefore subspace $B_1 = G \cap B' = G \cap \operatorname{Ker} p_1^\perp$ has a countable codimension in G. Let $\{e_k\}$ be the algebraic basis of the complement $G \cap B'$ in G and H is a subspace spanned by $\{e_k\}$, then $e_k \in G \cap \varphi$ and $H \subset \varphi$. The subspace H is closed in φ, since, due to the known properties of the space φ each of its vector subspaces is closed (topology of space φ is the strongest locally convex topology). Moreover, H is isomorphic to the space φ, and has a topological complement in φ [49], and therefore, in the induced topology from the space φ it is a strict (LB)-space. It also follows that G is isomorphic to the space $B_1 \times \varphi$ and is a strict (LB)-space.

We can give another proof of this fact. Namely, as is known, the quotientspace $((B' \times \varphi)/G, \mathfrak{T}/G) = \lim\limits_{\to} (\operatorname{Ker} p_n^\perp, \|\cdot\|_n)/(G \cap \operatorname{Ker} p_n^\perp)$. The strictness of this inductive limit can be proved by reasoning similar to those given higher. Then, by virtue of the first part of Theorem 2 from [142] it turns out that $(G, \beta(F', F) \cap G) = s \cdot \lim\limits_{\to} (G \cap \operatorname{Ker} p_n^\perp, \|\cdot\|_n)$.

2.3.2 Quojections of continuous functions and measures

Space of continuous functions $C(T)$. Let T be a separated locally compact space countable at infinity, i.e. T is represented as union of a sequence of compact sets $\{T_n\}$ and each a compact subset K is contained in some T_{n_0}. We can assume that for each $n \in \mathbb{N}$ the set $T_n \subset \operatorname{int} T_{n+1}$ and $T = \bigcup\limits_{n \in \mathbb{N}} T_n$. We denote by $C(T)$ (respectively $C_R(T)$) linear space of continuous complex (resp. real) functions on T with the topology of compact convergence on T. This topology can be defined by the sequence of seminorm

$$p_n(f) = \max\{|f(t)|; \ \ t \in T_n\}. \tag{2.3.1}$$

Let us introduce the following notation $V_n = \{f \in C(T);\ p_n(f) \leq 1\}$, $B = \{f \in C(T);\ \sup\{|f(t)| : t \in T\} \leq 1\}$. Let us prove that for each $n \in \mathbb{N}$ the equality $V_n = B + \operatorname{Ker} p_n$ is true. Let $f \in V_n$. By virtue of Urisohn's continuation theorem ([30], p. 47), the restriction of functions f on T_n can be continued on T so that the continuation of the function $f(t)$ will be bounded on T by the number 1. Let $f_1(t)$ be one of these continuations. Then $f_2 = f - f_1$ is equal to zero on T_n and therefore $f_2 \in \operatorname{Ker} p_n$. Therefore, $f = f_1 + f_2$, where $f_1 \in B$ and $f_2 \in \operatorname{Ker} p_n$. Inverse inclusion is obvious, therefore $C(T)$ is a quojection and the quotient spaces $C(T)/\operatorname{Ker} p_n$ are Banach spaces by the norm

$$\widehat{p}_n(k_n f) = p_n(f),$$

where $k_n : C(T) \to C(T)/\operatorname{Ker} p_n$ is the canonical map. Elements of quotient space $C(T)/\operatorname{Ker} p_n$ are the classes of all extensions of restrictions to T_n continuous functions defined on T. These classes are uniquely determined by restrictions to T_n of continuous functions. Therefore, the space $(C(T)/\operatorname{Ker} p_n, \widehat{p}_n)$ is isometric to the Banach space $C(T_n)$, that is, the space of continuous functions on the compact set T_n. Up to the isomorphism we can write that

$$C(T) = s \cdot \lim_{\leftarrow} \pi_{nm}(C(T_m)),$$

where $\pi_{nm} : C(T_m) \to C(T_n)$ is restriction operator on T_n of a continuous function defined on T_m.

In particular, the space $C(\Omega), \Omega \subset R^l$ is an open set, is a quojection. Moreover, in ([177], Chapter 3, Section 2) it is proved that the space of k-times continuously differentiable functions $C^k(\Omega)$ is isomorphic to the space $(C^k(I^l))^N$, where $I = [0,1]$.

Measure space $M(T)$**.** Let again T be locally compact space from the previous example. $\mathcal{K}(T)$ denotes the linear subspace in the space $C(T)$, consisting of continuous complex functions with a compact support. Let K be a compact subset of T. Let $\mathcal{K}(T, K)$ denote the set of those functions $x \in \mathcal{K}(T)$, whose supports are contained in K. Each of the sets $\mathcal{K}(T, T_n)$ is subspace in $\mathcal{K}(T)$ and the space $\mathcal{K}(T)$ is the union of spaces $\mathcal{K}(T, T_n)$, i.e. $\mathcal{K}(T) = \bigcup\limits_{n \in \mathbb{N}} \mathcal{K}(T, T_n)$. If each space $\mathcal{K}(T, T_n)$ is endowed by the norm

$$\|f\|_n = \max\{|f(t)|;\ \ t \in T_n\},$$

then it becomes a Banach space with a unit ball

$$U_n = \{f \in \mathcal{K}(T, T_n);\ \ \|f\|_n \le 1\}.$$

It is obvious that $\|\cdot\|_n$ is a restriction of the seminorm (2.3.1) to $\mathcal{K}(T, T_n)$.

The space $\mathcal{K}(T)$ can be endowed with the topology of an inductive limit $\mathfrak{T}$ with respect to identical embeddings $I_{nm} : \mathcal{K}(T, T_n) \to \mathcal{K}(T, T_m) \quad (n \le m)$.

Obviously, for each $n \in \mathbb{N}$ the norm $\|\cdot\|_{n+1}$, of the space $\mathcal{K}(T, T_{n+1})$ induces on $\mathcal{K}(T, T_n)$ the topology of the norm $\|\cdot\|_n$, i.e. in the $\mathfrak{T}$ topology the space $\mathcal{K}(T)$ is a strict (LB)-space, $\mathcal{K}(T) = s \cdot \lim\limits_{\rightarrow} \mathcal{K}(T, T_n)$.

By virtue of Proposition 2.2.2 on $\mathcal{K}(T)$ there exists $\mathfrak{T}$ - continuous norm which induces on every $\mathcal{K}(T, T_n)$ a norm topology $\|\cdot\|_n$. As such a norm we can take the norm defined by the equality

$$\|f\| = \max\{|f(t)|; \ \ t \in T\}, \ \ f \in \mathcal{K}(T).$$

If $U = \{f \in \mathcal{K}(T); \ \|f\| \leq 1\}$, then $U = \bigcup\limits_{n \in \mathbb{N}} U_n$ and $U \cap \mathcal{K}(T, T_n) = U_n$. It should be noted that U coincides with $B \cap \mathcal{K}(T)$, where B is the set from the previous example. Moreover, the U is neighborhood on $\mathcal{K}(T)$, and B is a bounded set in $C(T)$. From Corollary 1 of Proposition 2.2.2 it follows that the normed space $(\mathcal{K}(T), \|\cdot\|)$ is not complete, but $\mathcal{K}(T, T_n)$ are complete subspaces of $(\mathcal{K}(T), \|\cdot\|)$ for each $n \in \mathbb{N}$. The completion of the space $(\mathcal{K}(T), \|\cdot\|)$ is space of continuous on T functions $f(t)$ tending to 0 when t tends to the infinitely distant point at infinity ([30], p. 73).

According to [30], the Radon measure on a locally compact space T is a linear functional μ on the linear space $\mathcal{K}(T)$ satisfying the following condition: for any compact set $T_n \subset T$ the restriction of μ to subspace $\mathcal{K}(T, T_n)$ is continuous in the norm topology $\|\cdot\|_n$. The space of all Radon measures on T is denoted by $M(T)$. From Theorem 2.3.2 it follows that the Radon measure on T is a linear functional on $\mathcal{K}(T)$, continuous (even sequentially continuous) in the inductive limit topology $\mathfrak{T}$, therefore the space $M(T) = \mathcal{K}(T)'$.

A measure μ is said to be bounded in the Bourbaki sense if there exists $C > 0$ such that for any function $f \in \mathcal{K}(T)$

$$|\langle \mu, f \rangle| \leq C\|f\|.$$

Thus, the boundedness of the measure μ means that μ belongs to the dual to the normed space $(\mathcal{K}(T), \|\cdot\|)$. We denote this dual space by $M^1(T)$.

As follows from Theorem 2.3.2, the space $M(T)$, endowed with a strong topology of the duality $\langle \mathcal{K}(T), M(T) \rangle$, is a quojection. Let's give specification of this theorem in the case of the space $M(T)$.

Homothetic images of sets U_n form a fundamental sequence of bounded sets in $\mathcal{K}(T)$. Therefore polars U_n^0 generate a strong topology $\beta(M(T), \mathcal{K}(T))$ in $M(T)$. As we know $U_n^0 = U^0 + \operatorname{Ker} p_n^{\perp}$, where $U^0 = \{\mu \in M(T); \ \sup\{|\langle f, \mu \rangle|; \|f\| \leq 1\} \leq 1\}$ is $\sigma(M(T), \mathcal{K}(T))$ compact set. From here it is clear that the set U^0 contains measures that are bounded in the Bourbaki sense with number 1, i.e. U^0

represents the unit ball of the space $M^1(T)$. The subspace $\mathcal{K}(T,T_n)^\perp$ contains the measures $\mu \in M(T)$ whose supports do not intersect with $\operatorname{int} T_n$, i.e.

$$\mathcal{K}(T,T_n)^\perp = \{\mu \in M(T);\ \ \operatorname{supp}(\mu) \cap \operatorname{int} T_n = \emptyset\}.$$

Indeed, let $\mu \in \mathcal{K}(T,T_n)^\perp$ and $t \in \operatorname{supp}(\mu) \cap \operatorname{int} T_n$. Then there exists a neighborhood S of the point t such that $S \subset \operatorname{int} T_n$. For this neighborhood there is a continuous function $g(t)$ with compact support contained in S such that $\langle g, \mu\rangle \neq 0$. The obtaining contradiction shows that $\operatorname{supp}(\mu) \cap \operatorname{int} T_n = \emptyset$.

Now let $\operatorname{supp}(\mu) \cap \operatorname{int} T_n = \emptyset$ and f be a function with compact support contained in T_n, then $f(t) = 0$ for $t \in T_n \setminus \operatorname{int} T_n$. Indeed, if for some $t_0 \in T_n \setminus \operatorname{int} T_n$ we have $f(t_0) \neq 0$, then there exists a number $a > 0$ and a neighborhood S of the point t_0 such that $|f(t)| \geq a$, when $t \in S$. Therefore we obtain that $\operatorname{int} T_n \cup S \not\subset T_n$, i.e. the support of function f is not contained in T_n. This means that $f(t) = 0$, when $t \in T_n \setminus \operatorname{int} T_n$. Hence, by virtue of Proposition 8 ([30], p. 86), we find that $\langle f, \mu\rangle = 0$, i.e. $\mu \in \mathcal{K}(T,T_n)^\perp$.

The Minkowski functional p_n for U_n^0 have the following form:

$$p_n(\mu) = \sup\{|\langle f,\mu\rangle|;\ \ f \in U_n\} = \sup\{|\langle f,\mu\rangle|;\ \ f \in U \cap \mathcal{K}(T,T_n)\}$$

and the sequence $\{p_n\}$, by virtue of the above, generates strong topology in $M(T)$. By virtue of statement a) of Theorem 2.3.2, we have that

$$(M(T), \beta(M(T), \mathcal{K}(T))) = s \cdot \lim_{\leftarrow} M(T)/\operatorname{Ker} p_n = s \cdot \lim_{\leftarrow} M(T)/\mathcal{K}(T,T_n)^\perp,$$

with respect to canonical mappings $\pi_{nm} : M(T)/\mathcal{K}(T,T_m)^\perp \to M(T)/\mathcal{K}(T,T_n)^\perp$. Quotient space $M(T)/\mathcal{K}(T,T_n)^\perp$ is a Banach space in the quotient topology generated by the norm $\widehat{p}_n(k_n\mu) = p_n(\mu)$, where $k_n : M(T) \to M(T)/\mathcal{K}(T,T_n)^\perp$ is the canonical mapping. The space $M(T)\mathcal{K}(T,T_n)^\perp$ is isometric to the Banach space of measures on T, whose supports are contained in T_n. This space we denote by $M(T,T_n)$. Therefore, we have

$$M(T) = s \cdot \lim_{\leftarrow} M(T,T_n).$$

In the work [176] it was proved that the space of measures $M(T)$, where T is locally compact space countable at infinity, is isomorphic to the space $(C[0,1]')^N$, i.e. the space $M(T)$ in this case is a trivial quojection. It was also proved there that this result is valid for the space $M(V)$, where V is a non-compact locally compact manifold, countable at infinity.

Proposition b) of Theorem 2.3.2 makes it possible to describe the strong adjoint to the space $M(T)$ using the second conjugate Banach spaces $\mathcal{K}(T,T_n)$. The representations are valid

$$(M(T)', \beta(M(T)', M(T))) = s \cdot \lim_{\rightarrow} \mathcal{K}(T,T_n)'' = s \cdot \lim_{\rightarrow} (M(T)/\mathcal{K}(T,T_n)^\perp)'.$$

It is well known that the dual to the space $C(T)$ is space of Radon measures with compact support $M_c(T)$. By Theorem 2.3.2, for the strongly dual space $M_c(T)$ the following representation holds:

$$(M_c(T), \beta(M_c(T), C(T))) = s \cdot \lim_{\rightarrow} \operatorname{Ker} p_n^{\perp},$$

where the seminorms p_n are defined by the equality (2.3.1). As noted above, $\operatorname{Ker} p_n^{\perp} = (C(T)/\operatorname{Ker} p_n)' = C(T_n)'$. Therefore the last space is isomorphic to the space of measures on T_n. Therefore,

$$(M_c(T), \beta(M_c(T), C(T))) = s \cdot \lim_{\rightarrow} M(T_n) = s \cdot \lim_{\rightarrow} M(T, T_n),$$

where the last space is defined above.

As noted above, the Radon measure on T is continuous linear form on $\mathcal{K}(T)$. In ([30], p. 72) it is proved that the Lebesgue measure is not bounded on R by the norm $\|\cdot\|$, that is, does not belong to the space $(\mathcal{K}(R), \|\cdot\|)'$. But for any Radon measure μ there is a norm $\|\cdot\|_\mu$, inducing on each $\mathcal{K}(T, T_n)$ norm topology $\|\cdot\|_n$ such that $\mu \in (\mathcal{K}(T), \|\cdot\|_\mu)'$, that is μ is bounded by the norm $\|\cdot\|_\mu$. For the Lebesgue measure of such a norm we can take the norm on $\mathcal{K}(R)$ defined by the equality

$$\|f\|_\mu = \max\{(1+t^2)|f(t)|;\ \ t \in R\}, \quad f \in \mathcal{K}(R).$$

Indeed, the following relations are valid:

$$\sup\{|\langle f, \mu\rangle|;\ \|f\|_\mu \leq 1\} = \sup\left\{\left|\int_{-\infty}^{\infty} f(t)dt\right|;\ \|f\|_\mu \leq 1\right\}$$

$$\leq \sup\left\{\left|\int_{-\infty}^{\infty} \frac{(1+t^2)f(t)dt}{(1+t^2)}\right|;\ \ \|f\|_\mu \leq 1\right\} \leq \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi,$$

i.e. the Lebesgue measure is bounded with respect to the norm $\|\cdot\|_\mu$, which induces on each $\mathcal{K}(R, T_n)$, where $T_n = [-n, n]$, the norm topology $\|\cdot\|_n$.

2.3.3 Quojections of locally summable functions

1. Space $L^p_{loc}(T, \mu)$ $(1 \leq p \leq \infty)$**.** Let again T be locally compact space, countable at infinity, $T = \bigcup\limits_{n\in\mathbb{N}} T_n$, where $T_n \subset \operatorname{int} T_{n+1}$ are compact sets and $\mu \in M(T)$. Collection all complex valued, measurable functions f, defined on T, for which for any $n \in \mathbb{N}$ the quantity

$$\|f\|_n = \begin{cases} \Big(\int\limits_{T_n} |f(t)|^p d\mu(t)\Big)^{1/p}, & 1 \leq p \leq \infty, \\ \operatorname{vrai}\sup\{|f(t)|;\ t \in T_n\}, & p = \infty, \end{cases} \tag{2.3.2}$$

is finite, is called the space of p-locally summable functions in T (for $p = 1$, this is the space of locally summable functions on T). It is Fréchet space and is denoted by $L^p_{loc}(T, \mu)$. If μ is the Lebesgue measure on R, then the indicated space is denoted by $L^p_{loc}(R)$. Let us prove that with a sequence of seminorms (2.3.2) it is a quojection. Indeed, it is easy to check that the quotient space $L^p_{loc}(T, \mu)/\operatorname{Ker}\|\cdot\|_n$ is isomorphic to the Banach space $L^p(T_n, \mu)$ of functions summable on T_n with respect to restriction μ on T_n. Obviously, in this case it is also true the equality

$$V_n = B + \operatorname{Ker}\|\cdot\|_n,$$

where $V_n = \{f \in L^p_{loc}(T, \mu);\ \|f\|_n \leq 1\}$ and

$$B = \left\{ f \in L^p_{loc}(T, \mu);\ \int_T |f(t)|^p d\mu(t) \leq 1 \right\}.$$

Therefore, $L^p_{loc}(T, \mu) = s \cdot \lim\limits_{\leftarrow} \pi_{nm}(L^p(T_m, \mu))$, where $\pi_{nm} : L^p(T_m, \mu) \to L^P(T_n, \mu) \quad (m \geq n)$ is the restriction operator. Therefore, for $1 < p < \infty$ the space $L^p_{loc}(T, \mu)$ is the projective limit of a sequence of reflexive Banach spaces $\{L^p(T_n, \mu)\}$ and by virtue of [65] it is totally reflexive, i.e. each of its quotient spaces is again reflexive.

A measurable function is called compactly supported in T, if it vanishes almost everywhere outside some compact set contained in T. The set of all compactly supported functions from $L^p_{loc}(T, \mu)$ is denoted by $L^p_0(T, \mu)$. The set of all finite functions that turn into zero almost everywhere outside some set $K \Subset T$ is denoted by $L^p_0(K, \mu)$. Obviously, $L^p_0(T, \mu) = \bigcup\limits_{n \in \mathbb{N}} L^p_0(T_n, \mu)$. Conjugate to the space $L^p_{loc}(T, \mu)$ $(1 \leq p < \infty)$ is identified with the space $L^q_0(T, \mu)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

In the strong topology, the space $L^q_0(T, \mu)$ is identified with strict inductive limit of an increasing sequence of Banach spaces $\{L^q_0(T_n, \mu)\}$, i.e.

$$(L^q_0(T, \mu), \beta(L^q_0(T, \mu), L^p_{loc}(T, \mu))) = s \cdot \lim_{\rightarrow} L^q_0(T_n, \mu)$$

relatively identical embeddings $\pi'_{nm} : L^q_0(T_n, \mu) \to L^q_0(T_m, \mu)$.

2. The space of p-locally summable double sequences. Let l^p_{loc} $(1 \leq p < \infty)$ denote the echelon space p-th order on the set of indices $N \times N$, defined by double sequences $a^{(n)}$, where

$$a^{(n)}_{ij} = \begin{cases} 1, & \text{when } i \leq n,\ j \text{ arbitrary}, \\ 0, & \text{at } i > n,\ j \text{ arbitrary}, \end{cases}$$

i.e. l^p_{loc} is the space of all double sequences $x = \{x_{ij}\}$, the product of which with any $a^{(n)}$ is summable to the p-power. This we call the space of p-locally summable doubles sequences. The topology of the space l^p_{loc} is given by the following sequence of seminorms:

$$p_n(x) = \left(\sum_{i=1}^{n} \sum_{j=1}^{\infty} |x_{ij}|^p \right)^{1/p}, \quad x = \{x_{ij}\} \in l^p_{loc}, \quad n \in \mathbb{N}.$$

The space l^p_{loc} is a quojection and a quotient space $l^p_{loc}/\operatorname{Ker} p_n$ is isomorphic to the Banach space l^p.

The double sequence $x = \{x_{ij}\}$ is called rowwise finite of order n, if $x_{ij} = 0$ for all $i > n$. The set of all such sequences from the space l^p_{loc} we denote by $l^p_0(n)$. Space $l^p_0 = \bigcup\limits_{n\in\mathbb{N}} l^p_0(n)$ will be called the space of row-finitely supported p-integrable double sequences. Conjugate to the space l^p_{loc} $(1 \le p < \infty)$ is identified with the space l^q_0 $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. In the strong topology, l^q_0 is a strict (LB)-space and $(l^q_0, \beta(l^q_0, l^p_{loc})) = s \cdot \lim\limits_{\to} l^q_0(n)$.

3. Let $\Omega \subset R^l$ be an open set and $\{\Omega_n\}$ be an increasing sequence of its compact subsets such that $\Omega_n \subset \operatorname{int} \Omega_{n+1}$ for each $n \in \mathbb{N}$ and $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$. Let $W^{p,k}_{loc}(\Omega)$ $(1 \le p < \infty,\ k \in \mathbb{N})$ denote the space of all real functions f, having generalized derivatives of order $s = (s_1, \ldots, s_l)$, where $|s| = \sum s_i \le k$, belonging to the Banach space $L^p(\Omega_n)$ for each $n \in \mathbb{N}$. Space $W^{p,k}_{loc}(\Omega)$ is a Fréchet space with a sequence of seminorms

$$\|f\|^{p,k}_n = \left(\sum_{|s|\le k} \int\limits_{\Omega_n} |f^{(s)}(t)| dt \right)^{1/p}, \quad f \in W^{p,k}_{loc}(\Omega), \quad k, n \in \mathbb{N}.$$

Let $U^{p,k}_n = \{f \in W^{p,k}_{loc};\ \|f\|^{p,k}_N \le 1\}$ and

$$B = \left\{ f \in W^{p,k}_{loc};\ \left(\sum_{s\le k} \int\limits_{\Omega} |f^{(s)}(t)|^p dt \right)^{\frac{1}{p}} \le 1 \right\}.$$

It is obvious that

$$U^{p,k}_n \subset B + \operatorname{Ker} \|\cdot\|^{p,k}_n \subset U^{p,k}_{n-1} \ \text{ for any } \ n \ge 2.$$

According to statement b) of Theorem 2.3.1, this means that the space $W^{p,k}_{loc}$ is quojection.

The function $f \in W^{p,k}_{loc}(\Omega)$ is called finite of order n, if it vanishes almost everywhere outside Ω_n. The set of all such functions from the space $W^{p,k}_{loc}(\Omega)$ we denote by $W_0^{p,k}(\Omega_n)$ and call it the space of p-summable finite functions of order n. The space $W_0^{p,k}(\Omega) = \bigcup_{n\in\mathbb{N}} W_0^{p,k}(\Omega_n)$ will be called the space of all p-summable finite functions. In the topology of the inductive limit, the space $W_0^{p,k}(\Omega)$ is a strict (LB)-space and we have the representation $W_0^{p,k}(\Omega) = s \cdot \lim\limits_{\rightarrow} W_0^{p,k}(\Omega_n)$. The strong dual space to the space $W_0^{p,k}(\Omega)$ is a quojection by virtue of Theorem 2.3.2 and it is true the representation

$$((W_0^{p,k}(\Omega))', \beta((W_0^{p,k}(\Omega))', W_0^{p,k}(\Omega))) = s \cdot \lim_{\leftarrow}(W_0^{p,k}(\Omega_n))'.$$

In the case $k = 0$ the last space coincides with the space $L^p_{loc}(\Omega)$. It is easy to verify that $W^{p,k}_{loc}(\Omega) = (W_0^{q,k}(\Omega))'$ $(1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1)$.

4. Consider the set $\mathfrak{M}_{\nu p}(R)$ $(1 \le p < \infty,\ \nu > 0)$ of all entire functions $g(z)$ of exponential type ν, whose restrictions to R belong to $L^p(R)$. It is known that $\mathfrak{M}_{\nu p}(R)$ is closed subspace of the space $(L^p(R), \|\cdot\|_p), \mathfrak{M}_{\nu_1 p}(R) \subset \mathfrak{M}_{\nu_2 p}(R)$, when $\nu_1 \le \nu_2$ and $\mathfrak{M}_p(R) = \bigcup_{\nu\ge 0} \mathfrak{M}_{\nu p}(R)$ is dense in $L^p(R)$ ([116], p. 245). In the topology of the inductive limit, the space $\mathfrak{M}_p(R)$ is a strict (LB)-space, and its strongly conjugate space is a quojection.

Numerous examples of quojections appear when representing topology of generalized functions space D'.

2.4 Strict Fréchet–Hilbert space

This section studies topological, geometric and structural properties of strict Fréchet–Hilbert spaces, i.e. of quojections, which are represented as strict projective limits of a sequence of Hilbert spaces. And also studies their strong dual strict (LH)-spaces, subspaces and quotient spaces of these spaces.

2.4.1 Representation of the topology of the strict Fréchet–Hilbert space and its strong dual space

The Fréchet space $(E, \mathfrak{T})$ is called the strict Fréchet–Hilbert space if its topology $\mathfrak{T}$ is generated by a sequence of hilbertian seminorm $p_n(x) = (x,x)_n^{1/2}$, i.e. seminorms generated semi-inner products $(\cdot,\cdot)_n$ and the space E is complete with respect to each seminorm $p_n(\cdot)$. We can assume that sequence $\{p_n\}$ is non-decreasing, i.e. $p_1(x) \le p_2(x) \le \cdots \le p_n(x) \le \cdots$ for all $x \in E$.

It is easy to verify that the completeness of the Fréchet spaces E with respect to the seminorm p_n is equivalent to the fact that the quotient space $E/\operatorname{Ker} p_n$ is a Banach space with respect to associated norm $\widehat{p}_n$. Therefore, in the case of the strict Fréchet–Hilbert space $(E, \mathfrak{T})$, quotient spaces $E/\operatorname{Ker} p_n$ are Hilbert spaces according to the norms $\widehat{p}_n$. Therefore, a strict Fréchet–Hilbert space is a strict projective limit of sequences of Hilbert spaces $\{(E/\operatorname{Ker} p_n, \widehat{p}_n)\}$ with respect to canonical mappings $\pi_{mn} : (E/\operatorname{Ker} p_m, \widehat{p}_m) \to (E/\operatorname{Ker} p_n, \widehat{p}_n)(n \leq m)$. Such class of spaces appeared for the first time in the report [194] when representing the topology of the space of generalized functions D' (see also Section 2.6). In [85], there were received interesting properties of these spaces and various algebras of its continuous endomorphisms. It was generalized the concept of the self-adjoint operator and the spectral representation such operators was obtained. Similar representations in the case of Hilbert space are extremely important in quantum mechanics. Note that this is achieved by a natural generalization concepts of orthogonality in Fréchet–Hilbert spaces, like in the same way as was done earlier in [191] in the case of countable-Hilbert spaces. In the works [31, 84, 173, 191] were found orthogonal complements of various subspaces in the metrizable and in the non-metrizable case. Similar issues are discussed in more detail in the review [132].

Here we will study topological, geometrical and structural properties of strict Fréchet–Hilbert spaces and their strong dual strict (LH)-spaces, as well as subspaces and quotient spaces of these spaces. A rich class of subspaces of strict Fréchet–Hilbert spaces with topological complements is established. Moreover, the LCS $(E, \mathfrak{T})$ is a topological sum of its closed subspaces G and H means that $E = G + H$, $G \cap H = \{0\}$ and the quotient space $(E/G, \mathfrak{T}/G)$ is isomorphic to the subspace $(H, \mathfrak{T} \cap H)$. This fact will be denoted below as $(E, \mathfrak{T}) = G \dot{+} H$. A closed subspace G of an LCS E is said to have topological complement if there is a closed subspace H such that $E = G \dot{+} H$.

Theorem 2.4.1. *Let $(E, \mathfrak{T})$ be a real or complex non-normable Fréchet space with non-decreasing sequence of seminorms $\{p_n\}$ generating the topology $\mathfrak{T}$. Then the following statements are equivalent:*

a) *$(E, \mathfrak{T})$ is a strict Fréchet–Hilbert space with the sequence of seminorms $\{p_n\}$, that are generated by the semi-inner products $(\cdot, \cdot)_n$, and E is complete with respect to p_n $(n \in \mathbb{N})$.*

b) *The strong dual $E'_\beta = s \lim\limits_{\rightarrow} H_n$ is a strict (LH)-space, where each Hilbert space H_n is spanned by the polar V_n^0 of the neighborhood $V_n = \{x \in E, p_n(x) \leq 1\}$ and H_n has a topological complement in E'_β.*

c) *For each $n \in \mathbb{N}$ the equality $(E, \mathfrak{T}) = (H_n, p_{n,H_n}) \dot{+} \operatorname{Ker} p_n$, holds, where (H_n, p_{n,H_n}) is a Hilbert subspace of the space E with respect to the restriction p_{n,H_n} of p_n to H_n. In particular, $(E, \mathfrak{T}) = s \lim\limits_{\leftarrow} (H_n, p_{n,H_n})$.*

Proof. a) $\Rightarrow$ b). As mentioned above, for each $n \in \mathbb{N}$, the quotient space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ is a Hilbert space with respect to the norm $\widehat{p}_n(k_n x) = p_n(x)$

and the inner product $\langle k_n x, k_n y\rangle_n = (x,y)_n$, where $k_n : E \to E/\operatorname{Ker} p_n$ is the canonical homomorphism. This Hilbert space we denote by H_n. As was indicated in Corollary 1 of Theorem 2.3.1, the space $k_n'(E/\operatorname{Ker} p_n, \widehat{p}_n)' = k_n'(H_n')$ is isomorphic to the Hilbert space $E'_{V_n^0}$, spanned by V_n^0, i.e. $E'_{V_n^0}$ is isomorphic to H_n', and hence to H_n. Further, π'_{nm} $(n \le m)$ is the identity imbedding of $k_n'(H_n)$ in $k_m'(H_m)$, and thus H_n is isomorphic to a subspace of H_m. In view of Theorem 2.3.1 it was indicated that $E'_{V_n^0} = \operatorname{Ker} p_n^{\perp}$, where $\operatorname{Ker} p_n^{\perp}$ is the weakly closed subspace E', orthogonal to $\operatorname{Ker} p_n$ with respect to $\langle E, E'\rangle$. From here we get, that $H_n' = \operatorname{Ker} p_n^{\perp}$ and $E'_\beta = s \lim\limits_{\to} H_n$. Now we prove that H_n has topological complement in E'_β. Let $G_{n,i}$ be topological complement of the subspace H_{n+i-1} in H_{n+i} $(i \in \mathbb{N})$ and let $G_k^{(n)} = \oplus_{i=1}^k G_{n,i}$. Let $G_n = s \cdot \lim\limits_{\to} G_k^{(n)}$ is a strict (LH)-space. It is not hard to verify that G_n is an algebraic complement of H_n in E', i.e. $E' = H_n + G_n$ and $H_n \cap G_n = \{0\}$. The induced topology of $\beta(E', E) \cap G_n$ on G_n coincides with the topology of the original strict (LH)-space. Due to completeness of strict (LH)-space we obtain that G_n is closed in E'_β and $E'_\beta = H_n \dot{+} G_n$.

b) $\Rightarrow$ c). Passing to the polar of E, we get from the equalities established above that $H_n \cap G_n = \{0\}$ and $E' = H_n + G_n$, i.e. that $E = H_n^{\perp} + G_n^{\perp}$ and $H_n^{\perp} \cap G_n^{\perp} = \{0\}$, where $H_n^{\perp}$ and $G_n^{\perp}$ are $\sigma(E, E')$ closed subspaces of E, orthogonal to H_n and G_n respectively in the sense of a dual pair $\langle E, E'\rangle$. The following equalities are valid up to isomorphism:

$$\begin{aligned}(E/H_n^{\perp}, \mathfrak{T}/H_n^{\perp}) &= (E''/H_n^{\perp}, \beta(E'', E,')/H_n^{\perp}) = (H_n', \beta(H_n', H_n)) \\ &= ((E'/G_n)', \beta((E'/G_n)', (E'/G_n))) \\ &= (G_n^{\perp}, \beta(E'', E') \cap G_n^{\perp})) = (G_n^{\perp}, \mathfrak{T} \cap G_n^{\perp}) \,.\end{aligned}$$

Indeed, the first and last are valid because E is reflexive, the second is valid in view of a property of the representation of the dual of the subspace H_n [65], the third is valid because H_n and G_n are mutually complemented subspaces of E'_β, and the fourth is valid in view of the representation of the strong dual of the quotient space of the (DF)-space E'_β [49]. This means that $(E, \mathfrak{T}) = H_n^{\perp} \dot{+} G_n^{\perp}$. Further, $H_n^{\perp} = \operatorname{Ker} p_n^{\perp 1} = \operatorname{Ker} p_n$ and $G_n^{\perp} = (E/G_n)' = H_n' = H_n$, i.e. $E = \operatorname{Ker} p_n \dot{+} H_n$ and H_n is closed complemented subspace of E. It also follows from the last equality that the restriction p_{n,H_n} of the seminorm p_n to H_n is generated with inner product, with which H_n is a Hilbert space.

c) $\Rightarrow$ a) is obvious. □

As is well known, a Hilbert space is isomorphic to its dual space. The analogous property cannot hold in the case of nonnormable strict Fréchet–Hilbert spaces, because the dual of a Fréchet space is metrizable if and only if it is normable. But a similar property for the strict Fréchet–Hilbert space $(E, \mathfrak{T})$ is the condition that in view of Theorem 2.4.1, the space $(E, \mathfrak{T})$ can be represented as a strict projective limit of a sequence of Hilbert spaces, and in the space E'_β can be represented as a strict inductive limit of the same sequence. Moreover, H_n is a closed complemented subspace both for the space $(E, \mathfrak{T})$, and the space E'_β (If necessary emphasize that H_n is a subspace of E (respectively E'_β), then we denote it by $H_{n,E}$ (respectively $H_{n,E'}$)). It should also be noted that the dual the space $E' = \bigcup\limits_{n\in\mathbb{N}} H_{n,E'}$ is everywhere dense in space $(E, \mathfrak{T})$.

Corollary 1. *Let* $(E, \mathfrak{T}) = s \cdot \lim\limits_{\to} H_n$ *be a strict* (LH)*-space. Then every closed Hilbert subspace* G *of the space* $(E, \mathfrak{T})$*, i.e. subspace contained in some* H_n*, has a topological complement in* $(E, \mathfrak{T})$*.*

Corollary 2. *Let* $(E, \mathfrak{T})$ *be a strict Fréchet–Hilbert space and* p *is a continuous seminorm on* E*. Then the subspace* $\operatorname{Ker} p$ *has a topological complement in* E *that is Hilbert subspace of the space* $(E, \mathfrak{T})$*.*

The proved theorem makes it possible to construct strict Fréchet–Hilbert spaces using Hilbert spaces.

Let H be a Hilbert space and let $\{H_n\}$ be an increasing sequence of its closed subspaces such that $F = \bigcup\limits_{n\in\mathbb{N}} H_n$ is everywhere dense in $(H, \|\cdot\|)$. We define on H the sequence of seminorms p_n by equality

$$p_n(x) = \sup\{|(x,y)|;\ y \in H_n \cap S\}, \quad x \in H, \quad n \in \mathbb{N},$$

where S is the unit ball of space H. This sequence generates a metrizable topology on H, which is weaker than the norms topology. Obviously, H cannot be complete in this topology. Let us denote the completion of the space H in this topology via E. We prove that E is a strict Fréchet–Hilbert space. Indeed, consider the strict inductive limit of sequences $\{H_n\}$. Let $F = s \cdot \lim\limits_{\to} H_n$. Its strong dual space F'_β by Theorem 2.4.1 is a strict Fréchet–Hilbert space. Since the sequence of bounded sets $\{H_n \cap S\}$ forms a fundamental sequence, then the sequence of seminorms $\{p_n\}$ generates the topology of space F'_β. Hence it follows that $E = F'_\beta$.

In particular, the space $\mathfrak{M}_2(R)$ from Section 2.3 is a strict (LH)-space, and its strong dual space is strict Fréchet–Hilbert space.

2.4.2 Examples of strict Fréchet–Hilbert spaces

1. Fréchet space $(E, \mathfrak{T}) = \prod\limits_{k\in\mathbb{N}} (X_k, \|\cdot\|_k)$, where $(X_k, \|\cdot\|_k)$ are real Hilbert spaces, with the sequence of seminorms

$$p_n(x) = \left(\sum_{k=1}^{n} \|x_k\|_k^2\right)^{1/2}, \quad x = \{x_k\} \in E, \quad n \in \mathbb{N},$$

is a strict Fréchet–Hilbert space. Indeed, the subspaces H_n of the spaces E and E', mentioned in the theorem 2.4.1 can be taken to be the space $\prod\limits_{k=1}^{n} X_k$, which can be identified with subspace of E and E'. With the norm p_{n,H_n}, this subspace is a complemented Hilbert subspace in E.

In particular, the spaces $\omega = R^N(G^N)$, $l^2 \times \omega$ and $(l^2)^N$ are Fréchet–Hilbert spaces. Moreover, the space ω characterized in the class of Fréchet–Hilbert spaces by what is represented in the form of a strict projective limit of the sequence of their finite-dimensional subspaces. Further, as noted in Subsection 2.3, the space $l^2 \times \omega$ is characterized by the fact that on the space there is a sequence of seminorms p_n such that $\dim \operatorname{Ker} \pi_{nm} = \dim \operatorname{CoKer} \pi'_{nm} < \infty$ for any $m \geq n$.

2. The space l^2_{loc} of locally in a square summable double sequences. l^2_{loc} is the space of all double sequences whose product with any $a^{(n)}$ from Subsection 2.3 is square-summable. This space will be called the space of locally square-summable double sequences. It is the strict Fréchet–Hilbert space with the sequence of seminorms

$$p_n(x) = \left(\sum_{i=1}^{n}\sum_{j=1}^{\infty} |x_{ij}|^2\right)^{1/2}, \quad x = \{x_{ij}\} \in l^2_{loc}, \quad n \in \mathbb{N}\,.$$

The space $l^2_0 = \bigcup\limits_{n\in\mathbb{N}} l^2_0(n)$ is called the space of row-finite double sequences. The dual space $(l^2_{loc})'$ is identified with the space l^2_0. Due to the above, the equalities are true

$$l^2_{loc} = \operatorname{Ker} p_n \dot{+} l^2_0(n)\,,$$
$$l^2_{loc} = s \cdot \lim_{\leftarrow} l^2_0(n)\,,$$
$$(l^2_{loc})' = l^2_0 \;\text{ and }\; (l^2_0, \beta(l^2_0, l^2_{loc})) = s \cdot \lim_{\rightarrow} l^2_0(n)\,.$$

3. The space $L^2_{loc}(T,\mu)$. Let in the example from Section 2.3, $p = 2$. Space $L^2_{loc}(T,\mu)$ with a sequence of seminorms

$$p_n(x) = \left(\int\limits_{T_n} |x(t)|^2 d\mu\right)^{1/2}, \quad x \in L^2_{loc}(T,\mu), \quad n \in \mathbb{N},$$

is a strict Fréchet–Hilbert space. As subspaces H_n mentioned in Theorem 2.4.1, we can choose Hilbert space $L^2(T_n, \mu)$. In particular, the strict Fréchet–Hilbert space is the space $L^2_{loc}(\Omega, dx)$, where dx is the Lebesgue measure on the open set $\Omega \subset R^l$.

4. Space $W_0^{2,k}(\Omega)'$ ($\Omega \subset R^l,\ k \in \mathbb{N}$). Let $\Omega \subset R^l$ be an open set and $\{\Omega_n\}$ be its increasing sequence of compact subsets such that $\Omega_n \subset \operatorname{int} \Omega_{n+1}$ ($n \in \mathbb{N}$) and $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$. Let $W_{loc}^{2,k}(\Omega)$ denote the space all real functions f, having generalized derivatives $f^{(s)}$ of order $s = (s_1, \ldots, s_l)$, where $|s| = \sum\limits_{i=1}^{l} s_i \le k$ belonging to the space $L^2(\Omega_n)$ for each $n \in \mathbb{N}$. The space $W_{loc}^{2,k}(\Omega)$ is a strict Fréchet–Hilbert space with a sequence of seminorms

$$p_{n,k}(f) = \left(\sum_{|s|\le k} \int_{\Omega_n} |f^{(s)}|^2 dx \right)^{1/2}, \quad f \in W_{loc}^{2,k}(\Omega) \quad (n \in \mathbb{N})\,.$$

Let $W_0^{2,k}(\Omega)$ be the collection of all compactly supported functions from $W_{loc}^{2,k}(\Omega)$, which vanish almost everywhere outside Ω_n. We call the space $W_0^{2,k}(\Omega) = \bigcup\limits_{n\in\mathbb{N}} W_0^{2,k}(\Omega_n)$ the space of all summable compactly supported functions from the square $W_{loc}^{2,k}(\Omega)$. In the topology of the inductive limit $W_0^{2,k}(\Omega)$ is a strict (LH)-space and has the representation $W_0^{2,k}(\Omega) = s \cdot \lim\limits_{\to} W_0^{2,k}(\Omega_n)$. The dual space of the space $W_0^{2,k}(\Omega)$ in the strong topology is a strict Fréchet–Hilbert space with a sequence of seminorms

$$q_{n,k}(f) = \sup \left\{ \left| \sum_{|s|\le k} \int_{\Omega_n} f^{(s)}(t)\varphi^{(s)}(t)dt \right|;\ \varphi \in W_0^{2,k}(\Omega_n) \cap S \right\},$$

where

$$S = \left\{ \varphi \in W^{2,k}(\Omega);\ \left(\sum_{|s|\le k} \int_{\Omega} \left|\varphi^{(s)}\right|^2 dt \right)^{1/2} \le 1 \right\}.$$

We have inequalities for $f \in W_0^{2,k}(\Omega)'$

$$\begin{aligned} q_{n,k}(f) &= \sup \left\{ \left| \sum_{|s|\le k} \int_{\Omega_n} f^{(s)}(t)\varphi^{(s)}(t)dt \right|;\ \varphi \in W_0^{2,k}(\Omega_n) \cap S \right\} \\ &\le \sup \left\{ \left(\sum_{|s|\le k} \int_{\Omega_n} \left|f^{(s)}(t)\right|^2 dt \right)^{1/2} \cdot \left(\sum_{|s|\le k} \int_{\Omega_n} \left|\varphi^{(s)}(t)\right|^2 dt \right)^{1/2}; \right. \end{aligned}$$

$$\left.\varphi \in W_0^{2,k}(\Omega_n) \cap S\right\} \le p_{n,k}(f)\,.$$

From this it follows that the strict Fréchet–Hilbert space is strong dual to the space $W_0^{2,k}(\Omega)$ can be wider, than the space $W_{loc}^{2,k}(\Omega)$. We also have the inclusion $\operatorname{Ker} p_{n,k} \subset \operatorname{Ker} q_{n,k}$. Do these spaces coincide at least for some Ω, are not known.

2.4.3 On complemented subspaces of strict Fréchet–Hilbert spaces

As is known [92], Hilbert spaces can be characterized in the class of Banach spaces by the condition that they have the complemented subspace property, i.e., each closed subspace of a Hilbert space has a topological complement. In view of [66] (see also [49]) this gives us that among Fréchet spaces it is only in the spaces ω and $l^2 \times \omega$ each closed subspace has a topological complement. These results (see also [103]) suggest that in an arbitrary strict Fréchet–Hilbert space there must be an extensive class of complemented subspaces. Here we present such a class of subspaces and prove that the strict Fréchet–Hilbert space characterized in Fréchet spaces by the condition that each subspace in this class has a topological complement.

Definition. A subspace G of an LCS $(E, \mathfrak{T})$ will be called prequotient subspace if for some continuous seminorm p on E and some closed subspace $\widehat{G}$ of the quotient space $E/\operatorname{Ker} p$ the equality $G = k_p^{-1}(\widehat{G})$ holds, where $k_p : E \to E/\operatorname{Ker} p$, is the canonical mapping of E onto $E/\operatorname{Ker} p$.

Obviously, the prequotient subspaces are closed. Examples of prequotient subspaces of an LCS are subspaces of the type $\operatorname{Ker} p$, where p is continuous seminorm on E. In particular, closed hyperspaces and subspaces of finite codimension are such subspaces. Further, each closed subspace G of a normed space E is a prequotient space, because $G = \operatorname{Ker} p$ for $p(x) = \sup\{|\langle x, x'\rangle|;\ x' \in G^\perp\}$, where $G^\perp$ is orthogonal to G with respect to the duality $\langle E, E'\rangle$. Examples of subspaces that are not prequotient will be given below.

Lemma 2.4.2. *Let $(E, \mathfrak{T})$ be a quojections and G be its closed subspace. Then the following statements are equivalent:*

a) *G is a prequotient subspace.*

b) *There exists a continuous norm on the quotient space E/G.*

c) *$G = \operatorname{Ker} p$ for some continuous seminorm p on E.*

Proof. a) $\Rightarrow$ b). By assumption, there exist a continuous seminorm p on E and a closed subspace $\widehat{G}$ of the quotient spaces $E/\operatorname{Ker} p$ such that $G = k_p^{-1}(\widehat{G})$.

Since E is a quojections and there exists a continuous norm on their quotient space $E/\operatorname{Ker} p$, it follows that (as is known [111]) $E/\operatorname{Ker} p$ is a Banach space. Consider the quotient space $(E/\operatorname{Ker} p)/\widehat{G}$. It is known that a quotient space of a quotient space of the Fréchet space $(E, \mathfrak{T})$ is isomorphic to a quotient space of $(E, \mathfrak{T})$. In our case, it can be proved that the quotient space $(E/\operatorname{Ker} p)/\widehat{G}$ under consideration is isomorphic to the quotient space E/G. Since $E/\operatorname{Ker} p$ is a Banach space, a continuous norm exists also on the quotient space.

b) $\Rightarrow$ c). Let $\|\cdot\|$- be a continuous norm on quotient space E/G, then the seminorm p on E defined by the equality $p(x) = \|k_G(x)\|$, where $k_G : E \to E/G$ is the canonical mapping, is continuous and satisfies the condition $\operatorname{Ker} p = G$.

c) $\Rightarrow$ a) is obvious. □

It should be noted that in [37] were given several more statements that are equivalent to statement c) of Lemma 2.4.2 for an arbitrary LCS. In particular, such are:

d) There exist a continuous seminorm on E such that the subspace G is closed in this seminorm.

e) For every continuous seminorm p_0 on G, there exists its extension p to E such that $\operatorname{Ker} p_0 = \operatorname{Ker} p$.

As indicated in [37], in each non-normable Fréchet space E there exists a subspace G such that E/G is isomorphic to the space ω and hence there is no continuous norm on it. Consequently, in each strict Fréchet–Hilbert space $(E, \mathfrak{T})$ there exist subspaces that are not prequotient subspaces. In particular, the Hilbert subspaces H_n of it mentioned in Theorem 2.4.1 are such subspaces. This follows from Lemma 2.4.2, since $E/H_n = \operatorname{Ker} p_n$ and there is no continuous norm on $\operatorname{Ker} p_n$. It should also be mentioned that the same subspaces $H_{n,E'}$ are prequotient subspaces in the strong dual space E'_β. Indeed, as we established in the proof of the implication a)$\Rightarrow$b) in Theorem 2.4.1, the subspace $H_{n,E'}$ has a topological complement G_n in E'_β, that is a strict (LH)-space. Hence, there exists a continuous norm on G_n, and thus on E'/H_n. This gives us that the subspace $H_{n,E'}$ is a prequotient in E'_β.

Theorem 2.4.3. *A Fréchet space $(E, \mathfrak{T})$ is a strict Fréchet–Hilbert space if and only if each prequotient subspace of the space $(E, \mathfrak{T})$ has a topological complement.*

Proof. *Sufficiency.* Let p be an arbitrary continuous seminorm on E, $\widehat{G}$ is an arbitrary closed subspace of quotient space $E/\operatorname{Ker} p$ and $G = k_p^{-1}(\widehat{G})$, where $k_p : E \to E/\operatorname{Ker} p$ is the canonical mapping. By assumption, the subspaces $\operatorname{Ker} p$ and G have topological complement in E; therefore, there exist closed subspaces M and N of the space E such that $E = \operatorname{Ker} p \dotplus M$ and $E = G \dotplus N$. Obviously,

$\operatorname{Ker} p \subset G$ and $\operatorname{Ker} p \cap N \subset G \cap N = \{0\}$. This implies that the restriction of k_p to N is an algebraic isomorphism of the subspace N on $k_p(N) \subset E/\operatorname{Ker} p$ and the equality $E/\operatorname{Ker} p = \widehat{G} + k_p(N)$ holds. As mentioned above, the quotient spaces $(E/\operatorname{Ker} p)/\widehat{G}$ and E/G are topologically isomorphic; therefore the subspaces N and $k_p(N)$ are isomorphic and $E/\operatorname{Ker} p = \widehat{G} \dot{+} k_p(N)$. Accordingly, each closed subspace of the Fréchet space $E/\operatorname{Ker} p$, on which there exists a continuous norm, has a topological complement. Since there exists a continuous norm on $E/\operatorname{Ker} p$, then $E/\operatorname{Ker} p$ is isomorphic to the Hilbert space H_p. Hence it is easy to see that there is a hilbertian seminorm q on E such that $\operatorname{Ker} p = \operatorname{Ker} q$ and the restriction of q to H_p is generated by an inner product with which H_p is a Hilbert space. If we repeat these arguments for each seminorm from a sequence $\{p_n\}$ generating the topology of the space $(E, \mathfrak{T})$, then we obtain that $(E, \mathfrak{T})$ is a strict Fréchet–Hilbert space.

Necessity follows from Corollary 2 of Theorem 2.4.1 and Lemma 2.4.2. □

Corollary 1. *Let $(E, \mathfrak{T})$ be a Fréchet–Hilbert space. A subspace G is prequotient if and only if G has a topological complement, which is Hilbert subspace.*

Indeed, by Corollary 2 of Theorem 2.4.1 the prequotient subspace has a topological complement, which is a Hilbert subspace. Conversely, if G has a topological complement that is Hilbert subspace, then on the quotient space E/G there exists continuous norm. Therefore G is a prequotient subspace of space E.

Corollary 2. *Let $(E, \mathfrak{T})$ be a Fréchet–Hilbert space, G is its prequotient subspace and H is its Hilbert subspace such that $G \cap H = \{0\}$. Then $G + H$ is again prequotient subspace of the space $(E, \mathfrak{T})$.*

Indeed, there is a continuous Hilbert seminorm p on E and the Hilbert subspace H_p of E such that $G = \operatorname{Ker} p$ and $(E, \mathfrak{T}) = G \dot{+} (H_p, p_{Hp})$, where H_p is a Hilbert space with respect to the norm p_{H_p}. Let us prove that $k_p(H) \subset (E/\operatorname{Ker} p, \widehat{p})$ is a closed subspace in it. If $h \in H$ and $k_p(h) = 0$, then $h \in \operatorname{Ker} p \cap H = \{0\}$, i.e. the restriction of p to H is the norm. Therefore, the spaces H and $k_p(H)$ are algebraically isomorphic and this isomorphism is realized by the mapping k_{p,H_p}, which is restriction of k_p on H_p. Let $\{x_k\} \subset k_p(H)$ and $x_k \to x$ in $E/\operatorname{Ker} p$ by the norm $\widehat{p}$. It is necessary to prove that $x \in k_p(H)$. Let $h_k = k_{p,H_p}^{-1}(x_k) \in H$ and $h = k_{p,H_p}^{-1}(x) \in H_p$, then by condition $p(h_k - h) = \widehat{p}(k_p h_k - k_p(h)) \to 0$, for $k \to \infty$. If $\{p_n\}$ is a sequence of seminorms satisfying conditions of statement c) of Theorem 2.4.1, then there exists $n_0 \in \mathbb{N}$ and $C_{n_0} > 0$ such that $p(x) \leq C_{n_0} p_n(x)$ for $x \in E$ and $n \geq n_0$. The restriction of each seminorm p_n $(n \geq n_o)$ to H_p generates the topology of space H_p. Indeed, this follows from the fact that p_{n,H_p} stronger than p_{H_p}, (H_p, p_{H_p}) is complete and is a closed subspace of the spaces (H_n, P_{n,H_n}) $(n \geq n_0)$, as proved when proving the implication a) $\Rightarrow$ b) of Theorem 2.4.1. From here it follows that $p_n(h_k - h) \to 0$ at $k \to \infty$ for each $n \geq n_0$, i.e. due to the closedness of H, we have that $h \in H$ and $k_p(h) \in k_p(H)$. The prequotientity of $G + H$ follows from the equality $G + H = k_p^{-1}(k_p(H))$. Moreover, the spaces (H, p_{H_p}) and $(k_p(H), \widehat{p}_{k_p(H)})$ are topologically isomorphic.

Proposition 2.4.4. *Every prequotient subspace G of a strict Fréchet–Hilbert space $(E, \mathfrak{T})$ is the strict Fréchet–Hilbert space. The converse is not true, i.e. each nonnormable Fréchet–Hilbert space has a subspace of the same type that is not prequotient subspace.*

Proof. It suffices to prove that each subspace $\operatorname{Ker} p_{n_0}$ is a strict Fréchet–Hilbert space, i.e. its strong dual space is a strict (LH)-space. Indeed, using the notation of Theorem 2.4.1 we have that $\operatorname{Ker} p_{n_0} = E/H_{n_0}$. Further,

$$\begin{aligned}(\operatorname{Ker} p'_{n_0}, \beta(\operatorname{Ker} p'_{n_0}, \operatorname{Ker} p_{n_0})) &= ((E/H_{n_0})', \beta((E/H_{n_0})', E/H_{n_0})) \\ &= (H_{n_0}^{\perp}, \beta(H_{n_0}^{\perp}, E/H_{n_0})) = s \cdot \lim_{\rightarrow} H_{n_0} \cap H_{n,E'},\end{aligned}$$

since the quotient space E/H_{n_0} is again a reflexive quojection. It should be noted that the subspace $s \cdot \lim\limits_{\rightarrow} H_{n_0} \cap H_{n,E'}$ of space E'_β is nothing more than G_{n_0} in topology $\beta(E', E) \cap G_{n_o}$.

In the subspace ω, we consider a subspace G such that $g = \{g_n\} \in G$ are defined by

$$g_n = \begin{cases} 0\,, & \text{at } n = 2k, \\ a \in R\,, & \text{at } n = 2k+1\,. \end{cases}$$

The subspace G is a strict Fréchet–Hilbert space, but it is not prequotient, because for an arbitrary continuous seminorms p on ω the subspaces G and $\operatorname{Ker} p$ are distinct. The rest of our assertion follows from the fact that there is no continuous norm on a nonnormable strict Fréchet–Hilbert space and thus it contains a subspace isomorphic to ω that is not a prequotient subspace. □

Below we will prove that strict Fréchet–Hilbert spaces nonisomorphic to ω and $l^2 \times \omega$, have closed subspaces that are not strict Fréchet–Hilbert spaces.

2.4.4 Structural properties of strict Fréchet–Hilbert spaces. Properties of permanentness

A strict Fréchet–Hilbert space is a Montel space if and only if it is isomorphic to the nuclear space ω. This follows from the fact that a quojection is a Montel space if and only if it is isomorphic to ω [193]. On the other hand, each nuclear Fréchet space can be regarded as a closed subspace of the product of a sequence of separable Hilbert spaces l^2. Consequently, a closed subspace of the strict Fréchet–Hilbert space is not in general a strict Fréchet–Hilbert space. However, as mentioned above, each subspace of the spaces ω and $l^2 \times \omega$ is again a strict Fréchet–Hilbert space.

Proposition 2.4.5. *A quotient space of a strict Fréchet–Hilbert space is a strict Fréchet–Hilbert space.*

Proof. Let $(E, \mathfrak{T}) = s{\cdot}\lim\limits_{\leftarrow} H_n$ be a strict Fréchet–Hilbert space, and G be its closed subspace. If there is a continuous norm on E/G, then G is prequotient subspace in view of Corollary 1 of Theorem 2.4.3. If there is no continuous norm on the quotient space E/G, then it is a nonnormable quojection. Let us show that it is a strict Fréchet–Hilbert space. It is enough to prove that space $((E/G)', \beta((E/G)', E/G))$ is a strict (LH)-space. Indeed, $(E', \beta(E', E)) = s \cdot \lim\limits_{\rightarrow} H_n$ is a strict (LH)-space and hence

$$((E/G)', \beta((E/G)', E/G)) = (G^{\perp}, \beta(G^{\perp}, E/G)) = s \cdot \lim_{\rightarrow} H_n \cap G^{\perp}. \quad \square$$

It should be noted that the example constructed in [65] shows that the strict Fréchet–Hilbert space $E = (l^2)^N$ has a Montel subspace F such that the space $F^{\perp} \subset E'$ in the induced topology $\beta(E', E) \cap F^{\perp}$ is not a strict (LH)-space. By Proposition 2.4.5, the space $(F^{\perp}, \beta(F^{\perp}, E/F))$ is a strict (LH)-space. It will be proved below that arbitrary nonnormable strict Fréchet–Hilbert spaces not isomorphic to ω nor $l^2 \times \omega$ have such subspaces.

Theorem 2.4.6. *Each nonnormable separable strict Fréchet–Hilbert space $(E, \mathfrak{T})$ is isomorphic to only one space of the sequences ω, $l^2 \times \omega$ or $(l^2)^N$. In particular, the spaces l^2_{loc} and $L^2_{loc}(R)$ are isomorphic to the space $(l^2)^N$.*

Proof. From the Proposition 2.4.5 it follows that $E'_\beta = s \cdot \lim\limits_{\rightarrow} H_n$, where H_n are the Hilbert subspaces of the space E'_β. According to a proposition in ([52], Section 6) it suffices to prove that the space E'_β is isomorphic to the topological direct sum $\oplus_{n \in \mathbb{N}} F_n$, where F_n is a complemented subspace of E'_β. Indeed, as F_1 we take H_1, and as F_n $(n \geq 2)$ we take a complement of H_{n-1} to H_n. It is not hard to see that the identical mapping I of space $E'_\beta = s \cdot \lim\limits_{\rightarrow} H_n$ onto $\oplus_{n \in \mathbb{N}} F_n$ is linear and maps bounded sets into bounded sets, i.e. I is continuous. In view of the open mapping theorem for strict (LB)-spaces we get that these spaces are isomorphic, i.e. $E'_\beta = \oplus_{n \in \mathbb{N}} F_n$. Therefore, $(E, \mathfrak{T}) = (E'', \beta(E'', E')) = \prod\limits_{n \in \mathbb{N}} F'_n$.

Let us now consider the following three mutually exclusive cases: a) $\dim F_n < \infty$ for each $n \in \mathbb{N}$, b) $\dim F_1 = \infty$ and $\dim F_n < \infty$ for each $n \geq 2$. c) $\dim F_n = \infty$ for each $n \in \mathbb{N}$.

In the case a) the space E'_β is isomorphic to the space φ of all finite sequences and hence E is isomorphic to the space ω. In case b) E is isomorphic to space $l^2 \times \omega$. In case c) the space E is isomorphic to $(l^2)^N$. It should be mentioned that the general case reduces to one of the cases a) - c).

In particular, due to the separability of the spaces l^2_{loc} and $L^2_{loc}(R)$, they are isomorphic to the space $(l^2)^N$. □

Corollary. *Let $(E, \mathfrak{T})$ be a separable strict Fréchet–Hilbert space. Then the following statements hold:*

a) $(E, \mathfrak{T})$ *has an unconditional basis.*

b) $(E, \mathfrak{T})$ *has a nuclear Kethe subspace, i.e. subspace with continuous norm and basis if and only if it is isomorphic to the space* $(l^2)^N$.

c) *Every subspace and quotient space of the space* E'_β *is a strict* (LH)*-space if and only if when* $(E, \mathfrak{T})$ *is isomorphic to the space* ω *or* $l^2 \times \omega$.

Proof. a) follows from Theorem 2.4.6 and the proposition, proved in ([52], Section 6). b) follows from Theorem 2.4.6 and the results works [15]. Due to [49], the space ω, $l^2 \times \omega$ and their strongly adjoint spaces φ and $l^2 \times \varphi$ have the complemented subspace property, i.e. every closed subspace of these spaces has a topological complement and subspace and quotient spaces of the spaces φ and $l^2 \times \varphi$ are isomorphic to these spaces. □

2.4.5 Orthogonality in the Fréchet–Hilbert spaces

Let the topology of the Fréchet–Hilbert space E be given by non-decreasing sequence of hilbertian seminorms $\{p_n\}$, i.e. each seminorm p_n is generated by the semi-inner product $(\cdot, \cdot)_n$. In particular, such are nuclear Fréchet spaces, countable Hilbert spaces and strict Fréchet–Hilbert spaces. For such Fréchet–Hilbert spaces, it is naturally defined the concept of orthogonality: elements $x, y \in E$ are called orthogonal with respect to the inner product $(\cdot, \cdot)_n$, if $(x, y)_n = 0$. This fact is denoted as $x \perp^n y$. The elements $x, y \in E$ are called orthogonal in E, if $(x, y)_n = 0$ for each $n \in \mathbb{N}$. This fact in further is denoted as $x \perp y$.

Orthogonality defined in this way has some properties similar to orthogonality in Hilbert spaces. For example, $x \perp x$ if and only if $x = 0$. $x \perp y$ for each $y \in E$ if and only if $x = 0$.

Let $M \subset E$ be a bob-empty set. We will write $M_n^\perp = \{x \in E;\ x \perp^n m$ for all $m \in M\}$ and $M^\perp = \cap_{n \in \mathbb{N}} M_n^\perp$. It is obvious that the sets $M_n^\perp$ and $M^\perp$ are closed subspaces in E. If M is a closed subspace, then $M^\perp$ is called its orthogonal complement and denoted by $E = M \oplus M^{\perp}$. The problem naturally arises, when the space E can be represented as a sum of subspaces M and $M^{\perp}$, i.e. which subspace M has orthogonal complement $M^{\perp}$. In the case of countable Hilbert spaces orthogonal subspaces were found and characterized in [191], and in the case of strict Fréchet–Hilbert spaces they were defined and studied in [84, 133, 201]. For non-metrizable locally convex spaces, similar problems were considered in [31, 72, 173].

According to [191] a set G has property (H) in a Fréchet–Hilbert space E with a sequence of hilbertian seminorms $\{\|\cdot\|_n\}$ if for each $x \in E$ in G there is a sequence $\{g_k\}$ such that

$$\lim_{k\to\infty} \|x + g_k\|_n = \inf\{\|x + g\|_n;\ g \in G\}$$

for all $n \in \mathbb{N}$. It is also proved that a closed subspace G of a Fréchet–Hilbert space has an orthogonal complement $G^{\perp}$ if and only if it has property (H).

Let E be a countable Hilbert space with an increasing sequence of norms $\{\|\cdot\|_n\}$, which are generated by the inner products $\{(\cdot,\cdot)_n\}$. Let E_n denote the completion of the space E with respect to the norm $\|\cdot\|_n$. Obviously, E_n is a Hilbert space. From the completeness of the space E it follows that

$$E = \bigcap_{n\in\mathbb{N}} E_n\,. \tag{2.4.1}$$

In what follows we will need to consider some elements of the space E as elements of the corresponding Hilbert spaces E_n. In cases where this may lead to misunderstanding, we will write $\overset{(n)}{x}$ instead of x. Thus, in the case of a countable Hilbert space E, the elements $\overset{(1)}{x}, \overset{(2)}{x}, \ldots, \overset{(n)}{x}, \ldots$ is the same element $x \in E$ of different spaces E_n.

Let G be a subspace of E. We denote the closure of G in E_n by G_n, then the following representation holds:

$$G = \bigcap_{n\in\mathbb{N}} G_n\,.$$

By virtue of the Beppo-Levi theorem, for each $n \in \mathbb{N}$ the equality holds

$$H_n = G_n \oplus G_n^{\perp}\,, \tag{2.4.2}$$

where $G_n^{\perp}$ is the orthogonal complement of G_n to H_n, defined in a known way.

Let $f \in E$, then by (2.4.1) for all $n \in \mathbb{N}$ $\overset{(n)}{f}$ can be uniquely represented in the form

$$\overset{(n)}{f} = g_n + y_n\,,$$

where $g_n \in G_n$ and $y_n \in G_n^{\perp}$.

According to [19], a subspace G has property (C) if for any $f \in E$ there is a $g \in G$ such that for all $x \in \mathbb{N}$ $\overset{(n)}{g} = g_n$, where g_n are defined above. It is also proved that a closed subspace G of a countable Hilbert space E has an orthogonal complement $G^{\perp}$ if and only if it has property (C).

Let us now present a sufficient condition for the closed subspace G of the strict Fréchet–Hilbert space E to have an orthogonal complement $G^{\perp}$ in E.

Proposition 2.4.7. *Let in the notation of Theorem* 2.4.1 $E = s \cdot \lim\limits_{\leftarrow} H_n$ *be strict Fréchet–Hilbert space with a sequence of seminorms* $\{p_n\}$ *and* G *is a closed subspace satisfying the following conditions:*

a) $G \cap H_n = G_n$ *is a closed subspace in the Hilbert space* H_n.

b) *For each* $m \geq n$ *the equality* $\pi_{nm,G_m} \circ P_m = P_n \circ \pi_{nm}$ *is true, where* π_{nm,G_m} *is the restriction of* π_{nm} *on* G_m *(we identify* G_m *with* $k_m(G_m)$*),* P_m *is projection of the Hilbert space* H_m *on* G_m*, i.e. the following diagram is commutative*

$$\begin{array}{ccc} H_m & \xrightarrow{P_m} & G_m \\ \pi_{nm}\big\downarrow & & \big\downarrow\pi_{nm,G_m} \\ H_n & \xrightarrow{P_n} & G_n \end{array}$$

c) *The sequence* $\{G_n^{\perp}\}$ *of orthogonal complements* G_n *in* H_n *is increasing.*

Then G *has the orthogonal complement* $G^{\perp}$ *in* E*, i.e.* $E = G \oplus G^{\perp}$.

Proof. According to Theorem 2.4.1 each element $f \in E$ is identified with the sequence $\{k_n f\}$, where $\pi_{mn} \circ k_m f = k_n f$ for any $m \geq n$. One can also identify $k_n f$ with some element $h_n \in H_n$. Due to a), for each $n \in \mathbb{N}$ the following equality holds: $h_n = g_n + z_n$, where $g_n = P_n h_n \in G_n$, $z_n \in G_n^{\perp}$ and $(g_n, z_n) = 0$. Since by condition c) we have non-decreasing sequence of Hilbert spaces $\{G_n^{\perp}\}$, then this sequence defines a certain subspace, as a strict projective limit of the sequence $\{G_n^{\perp}\}$. It should be proved that it is the orthogonal complement $G^{\perp}$ of subspace G in E. For this it suffices to prove that the above sequences $\{g_n\}$ and $\{z_n\}$ determine the required elements $g \in G$ and $z \in G^{\perp}$, for which $f = g + z$ and $(g, z)_n = 0$ for every $n \in \mathbb{N}$. The subspace G is strict projective limit of a sequence of Hilbert spaces $\{(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})\}$. But spaces $(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})$, $(k_n(G), \widehat{p}_{n,k_n(G)})$ and $(k_n(G_n), \widehat{p}_{n,k_n(G_n)})$ are isometric. This means that G is a strict projective limit of sequence of Hilbert spaces $\{(k_n(G_n), \widehat{p}_{n,k_n(G_n)})\}$ relative to mappings π_{nm,G_m} $(n \leq m)$. It remains to prove that the sequence $\{k_n g_n\} = \{g_n\} = \{P_n \circ \pi_{nm} \circ k_m f\}$ generates an element of subspaces G, but this follows from condition b), since

$$P_n \circ \pi_{nm} \circ k_m f = g_n = \pi_{nm,G_m} \circ P_m \circ k_m f = \pi_{nm,G_m} \circ P_m h_m = \pi_{nm,G_m} g_m\,,$$

for $n \leq m$. Hence, there is an element $g \in G$ such that $g = \{g_n\}$, i.e. $g_n = k_n g$ for any $n \in \mathbb{N}$. Further, from the equality $h_m = g_m + z_m$ we obtain the equalities $k_m f_m = k_m g_m + k_m z_m = k_m g + k_m z_m$ and $\pi_{nm} \circ k_m f = \pi_{nm} \circ k_m g + \pi_{nm} \circ k_m z_m$. It follows from here that $k_n f = k_n g + \pi_{nm} \circ k_m z_m$, i.e. $\pi_{nm} \circ k_m z_m = k_n z_n$ for $n \leq m$. Therefore, the sequence $\{z_n\}$ defines element z from the projective limit of the sequence of Hilbert spaces $\{G_n^{\perp}\}$, i.e. from $G^{\perp}$. Therefore, we have also $(g_n, z_n)_n = \langle k_n g, k_n z\rangle_n = (g, z)_n = 0$ for each n and $f = g + z$. □

It should be noted that if in the Fréchet–Hilbert space E every finite-dimensional subspace has an orthogonal complement, then E is isomorphic to a Hilbert space. Really, if a subspace G has an orthogonal complement of $G^{\perp}$ in E, then for $f \in E$ there exists $g \in G$ and $z \in G^{\perp}$ such that $f = g + z$ and $(g, z)_n = 0$ for each $n \in \mathbb{N}$. Then such a subspace G is a Chebyshev subspace in E with respect to the metric (2.5.4), since for each $f \in E$ the only best approximation in G there will be its projection $g \in G$. By virtue of ([3], Theorem 8) we obtain that the balls of this metric $K_r(r \in R^+)$ will be strictly convex in E, i.e. the Minkowski functional q_r for K_r will be strictly convex norm for each $r \in]0, 1/2[$. And this is possible only in the case when the space E will be Hilbert. Again, by virtue of ([3], Theorem 8), it is sufficient require that every one-dimensional subspace has an orthogonal complement. An example of a one-dimensional subspace of the strict Fréchet–Hilbert space $L^2_{loc}(R)$, without orthogonal complement, is given in [84]. Very interesting results about strict Fréchet–Hilbert spaces were given by M. Poppenberg [129], D. Vogt [179], K. Piszczek [127], B. Dierolf [35], E. Uyanık, M. H. Yurdakul [171].

2.5 New metric on metrizable LCS

Theorem 2.5.1 ([82], p. 205)**.** *Let $(E, \mathfrak{T})$ be a metrizable LCS with a generating increasing sequence of seminorm $\{\|\cdot\|_n\}$. Then the topology $\mathfrak{T}$ of the space E is also can be given by metric*

$$d(x, y) = \sum_{n\in\mathbb{N}} \frac{\|x - y\|_n}{2^n(1 + \|x - y\|_n)}, \quad x, y \in E\,, \tag{2.5.1}$$

which is translation-invariant.

It was of great importance the construction by G. Albinus in [3] norm-like metrics. Translation-invariant metric d is called norm-like if the balls $K_r = \{x \in E;\ d(x, 0) \leq r\}$ are absolutely convex and the mapping of the positive semi-axis R^+ into itself, defined by the correspondence $t \to d(tx, 0)$ is strictly monotone for each $x \neq 0$.

Theorem 2.5.2 ([3], p. 181)**.** *On each metrizable LCS E there is an translation-invariant norm-like metric that generates its topology. In particular, if $\{\|\cdot\|_n\}$ is a generating sequence pairwise nonequivalent seminorms on E and $V_n = \{x \in E;\ p_n(x) \le 1\}$, then the metric defined by the equality*

$$d_1(x,y) = \begin{cases} p_1(x-y),\ \textit{when}\ x-y \in E\backslash V_1, \\ 2^n \max\{2-[p_{n+1}(x-y)]^{-1},\ 2p_n(x-y)\}, \\ \qquad\qquad\qquad\qquad \textit{when}\ \ x-y \in V_n\backslash V_{n+1}\ (n \in \mathbb{N}), \\ 0,\ \textit{when}\ x = y\,, \end{cases} \tag{2.5.2}$$

where $p_1(\cdot) = \|\cdot\|_1$, $p_{n+1}(\cdot) = \max\left\{\frac{2^n\|\cdot\|_{n+1}}{\|z\|_{n+1}},\ 2p_n(\cdot)\right\}$, $z \in E$ is any point defined by the equality $p_1(z) = 1$, is a translation-invariant norm-like metric on E, generating its topology.

For Minkowski functionals of q_r balls $K_r = \{x \in E;\ d_1(x,0) \le r\}$ metrics (2.5.2) the equalities are valid

$$q_r(x) = \begin{cases} r^{-1}p_1(\cdot),\ \text{when}\ 1 \le r < \infty, \\ p_{n+1}(\cdot),\ \text{when}\ r = 2^{-n}\ (n \in \mathbb{N}), \\ \max\{(2-2^n r)p_{n+1}(\cdot),\ \frac{2^{-n+1}}{r}p_n(\cdot)\}, \\ \qquad\qquad\qquad\qquad \text{when}\ \ r \in]2^{-n}, 2^{-n+1}[\ (n \in \mathbb{N})\,. \end{cases} \tag{2.5.3}$$

Theorem 2.5.3 ([5], p. 33)**.** *Let $(E,\mathfrak{T})$ be metrizable LCS and $\{p_n\}$ be a sequence of seminorms, generating the topology E. Then the metric defined by the equality*

$$d_2(x,y) = \sup_{n\in\mathbb{N}} \frac{p_n(x-y)}{2^n(1+p_n(x-y))}, \quad x,y \in E\,, \tag{2.5.4}$$

is a norm-like metric on E generating its topology.

For Minkowski functionals of q_r balls K_r metrics (2.5.4), the equalities are valid

$$\begin{aligned} q_r(x) &= \max_{n\le n_0} \frac{1-2^n r}{2^n r} p_n(x) \\ &= \max_{n<\frac{\ln 1/r}{\ln 2}} \frac{1-2^n r}{2^n r} p_n(x), \quad \text{when}\ \ r \in [2^{-n_0+1}, 2^{-n_0}[\,. \end{aligned} \tag{2.5.5}$$

Theorem 2.5.4 ([3], p. 185)**.** *Let (E,d) be LCS with metric* (2.5.1)*. If at least one of the seminorms $\|\cdot\|_j$ is not neither norm nor identically zero on E, then the metric d is not norm-like.*

Other metrics on a metrizable LCS can be find in [18, 76, 144]. Let's take a closer look at the metric from [146], which in turn is a modification of the metric from [76].

Let us denote by $[r]$ (resp. $\{r\}$) the integer (resp. fractional) part of the number $r \in R$. Let D be the set of all non-negative numbers r for which $\{r\}$ is represented as a finite binary fraction, i.e. $\{r\} = \sum\limits_{k\in\mathbb{N}} c_k(\{r\})2^{-k}$, where $c_k(\{r\})$ are the coefficients representations of the number $\{r\}$ equal to 0 or 1.

Theorem 2.5.5 ([146], p. 11)**.** *Let $(E, \mathfrak{T})$ be a metrizable linear topological space with a basis of closed, balanced neighborhoods of zero $\{V_n\}$, where $V_1 \neq E$ and $V_{n+1} + V_{n+1} \subset V_n$ $(n \in \mathbb{N})$. Then there exists translation-invariant metric which generates on E the topology $\mathfrak{T}$*

$$d(x, y) = \inf\{r \in D;\ x - y \in A_r\} \tag{2.5.6}$$

where the sets A_r have the following form

$$A_r = \underbrace{V_1 + \cdots + V_1}_{[r]} + \sum_{k\in\mathbb{N}} c_k(\{r\})V_{k+1}, \quad r \in D\,.$$

2.5.1 New metric on a metrizable LCS

We use the metric (2.5.6) to construct a new metric on a metrizable LCS.

Proposition 2.5.6. *Let $(E, \mathfrak{T})$ be a metrizable LCS with a generating sequence of seminorms $\{p_n\}$, where $p_1 \neq 0$, $2p_n(\cdot) \leq p_{n+1}$ $(n \in \mathbb{N})$, i.e. $V_1 \neq E$ and $2V_{n+1} \subset V_n$ $(n \in \mathbb{N})$, where $V_n = \{x \in E;\ p_n(x) \leq 1\}$. Then the equalities are true $A_r = [r]V_1 + A_{\{r\}}$ and $\overline{A}_r = K_r$ $(r \in D)$, where K_r is a closed ball of radius r of the metric* (2.5.6)*, and $\overline{A}_r$ means the closure of A_r. Next, for $r \in [2^{-n}, 2^{-n+1}[$ (respectively $r \in [1, \infty[$) Minkowski functionals q_r of balls K_r are equivalent to $p_{n+1}(\cdot)$ (respectively $p_1(\cdot)$).*

Proof. The family $\{A_r; r \in D\}$ is increasing, therefore the equality $K_r = \bigcap\limits_{\substack{s\in D,\\ s>r}} A_s$ is valid. Hence, K_r is an absolutely convex neighborhood such that $K_r \neq E (r \in R^+)$, i.e. the metric d is not bounded on E. Let us now show that $\overline{A}_r = K_r$ $(r \in R^+)$. The inclusion of $\overline{A}_r \subset K_r$ is obvious. Let $x \in K_r$, then $x \in A_{r+2^{-n}}$ $(n \in \mathbb{N})$. It is not difficult to find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the equalities are valid

$$[r + 2^{-n}] = [r],\ \{r + 2^{-n}\} = \{r\} + \{2^{-n}\} \quad \text{and}$$

$$A_{r+2^{-n}} = [r]V_1 + A_{\{r+2^{-n}\}} = [r]V_1 + A_{\{r\}} + A_{2^{-n}} = A_r + V_{n+1}\,.$$

From here it immediately follows that $x \in \overline{A}_r$. In particular, $A_{2^{-n}} = V_{n+1} = K_{2^{-n}}$.

Let now $s \in D_n = D \cap [2^{-n}, 2^{-n+1}[$, then $c_1(s) = \cdots = c_{n-1}(s) = 0$ and $c_n(s) = 1$, i.e. $A_s = \sum\limits_{k \geq n} c_k(s) V_{k+1}$. Therefore $V_{n+1} \subset K_r$ ($r \in [2^{-n}, 2^{-n+1}[$), i.e. the inequality $q_r(\cdot) \leq p_{n+1}(\cdot)$ is true. On the other side, for the mentioned s and r the inclusions hold

$$A_s = \sum_{k \geq n} c_k(s) V_{k+1} \subset \sum_{k \geq n} c_k(s) 2^{n-k} V_{n+1} \subset 2^n V_{n+1} \sum_{k \geq n} c_k(s) 2^{-k} = 2^n s V_{n+1}\,,$$

$$K_r = \bigcap_{s \in D,\, s > r} A_s = \bigcap_{s \in D_n,\, s > r} A_s \subset \bigcap_{s \in D_n,\, s > r} 2^n s V_{n+1} = 2^n r V_{n+1}\,,$$

those.

$$p_{n+1}(\cdot) \leq 2^n r q_r(\cdot) \leq 2 q_r(\cdot)\,.$$

If $r \in [1, \infty[$, then $V_1 \subset K_r \subset ([r]+1) V_1$. □

Corollary. *Let $(E, \mathfrak{T})$ be a metrizable LCS with a basis of decreasing neighborhoods of zero $\{V_n\}$ having the following form $V_n = 2^{-n+1} B + \operatorname{Ker} p_n$ ($n \in \mathbb{N}$), where B is closed, bounded, absolutely convex subset of the space E and p_n is Minkowski functional for V_n. Then for balls K_r of metric* (2.5.6) *the following equalities are true:*

$$K_r = \begin{cases} r V_1, & \textit{when } r \in [1, \infty[\,, \\ 2^n r V_{n+1}, & \textit{when } r \in [2^{-n}, 2^{-n+1}[\,. \end{cases}$$

Proof. Let $r \in D_n$, then

$$\begin{aligned} A_r &= \sum_{k \geq n} c_k(r) V_{k+1} = \sum_{k \geq n} c_k(r)(2^{-k} B + \operatorname{Ker} p_{k+1}) \\ &= B \sum_{k \geq n} c_k(r) 2^{-k} + \operatorname{Ker} p_{n+1} = rB + \operatorname{Ker} p_{n+1} \\ &= rB + \operatorname{Ker} p_{n+1} = 2^n r(2^{-n} B + \operatorname{Ker} p_{n+1}) = 2^n r V_{n+1} = K_r, \end{aligned}$$

due to the closedness of A_r. If $r \in [2^{-n}, 2^{-n+1}[$, then

$$\begin{aligned} K_r &= \bigcap_{s \in D,\, s > r} A_s = \bigcap_{s \in D_n,\, s > r} A_s = \bigcap_{s \in D_n,\, s > r} 2^n s V_{n+1} \\ &= 2^n r V_{n+1} = rB + \operatorname{Ker} p_{n+1}\,. \end{aligned}$$

Let now $r \in D \cap [1, \infty[$ and $\{r\} \in [2^{-n}, 2^{-n+1}[$, then

$$\begin{aligned} A_r &= [r]V_1 + A_{\{r\}} = [r]B + \operatorname{Ker} p_1 + \{r\}B + \operatorname{Ker} p_{n+1} \\ &= rB + \operatorname{Ker} p_1 = rV_1 = K_r \,. \end{aligned}$$

Similarly, it turns out that $K_r = rV_1$ for other $r \in [1, \infty[$. □

This corollary gave rise to the idea of proving the following theorem.

Theorem 2.5.7. *Let $(E, \mathfrak{T})$ be a metrizable LCS with a generating sequence of seminorms $\{p_n\}$, where $p_1 \neq 0$, $2p_n \leq p_{n+1}$ $(n \in \mathbb{N})$, i.e. $V_1 \neq E, 2V_{n+1} \subset V_n$ $(n \in \mathbb{N})$, where $V_n = \{x \in E;\ p_n(x) \leq 1\}$. Then by the family*

$$K_r = 2^{n-1} r V_n, \text{ where } r \in I_n = \begin{cases} [1, \infty[\,, & \text{when } n = 1, \\ [2^{-n+1}, 2^{-n+2}[\,, & \text{when } n \geq 2 \end{cases} \tag{2.5.7}$$

on E it is defined the metric $d(x, y) = \inf\{r \in R^+;\ x - y \in K_r\}$ with closed absolutely convex balls K_r and for its quasinorm $|\cdot|$ the equalities are valid

$$\begin{aligned} d(x,y) &= |x-y| \\ &= \begin{cases} p_1(x-y), & \text{when } x - y \in E \setminus \operatorname{int} V_1, \\ 2^{-n+1}, & \text{when } x - y \in \operatorname{int} V_n \setminus 2 \operatorname{int} V_{n+1} \quad (n \in \mathbb{N}), \\ 2^{-n} p_{n+1}(x-y), & \text{when } x - y \in 2 \operatorname{int} V_{n+1} \setminus \operatorname{int} V_{n+1} \quad (n \in \mathbb{N}), \\ 0, & \text{when } x - y = 0\,. \end{cases} \end{aligned} \tag{2.5.8}$$

Proof. To prove the triangle inequality it is enough prove that for any $r,\ s \in R^+$ inclusion $K_r + K_s \subset K_{r+s}$ is true. Consider the following three cases:

a) Let $r \in [2^{-n}, 2^{-n+1}[= I_{n+1}$ and $s \in [2^{-m}, 2^{-m+1}[$, where $1 \leq n \leq m$, $(n, m \in \mathbb{N})$. Then $r + s \in [2^{-n} + 2^{-m}, 2^{-n+1} + 2^{-m+1}[$. If $r + s \in I_{n+1}$, then $K_r + K_s = 2^n r V_{n+1} + 2^m s V_{m+1} \subset 2^n r V_{n+1} + 2^{m-1} s V_m \subset \cdots \subset 2^n r V_{n+1} = K_{r+s}$.

If $r + s \in I_n$, then $K_r + K_s = 2^n r V_{n+1} + 2^m s V_{m+1} \subset 2^{n-1} r V_n + 2^{n-1} s V_n = 2^{n-1}(r + s) V_n = K_{r+s}$.

b) Let $r \in [1, \infty[$ and $s \in [2^{-m}, 2^{-m+1}[= I_{m+1}$ $(m \in \mathbb{N})$, then $r + s \in I_1$ and $K_r + K_s = rV_1 + 2^m s V_{m+1} \subset rV_1 + sV_1 = K_{r+s}$.

c) The case $r, s \in I_1$ is trivial.

Let us now prove the formula (2.5.8). Let $x - y \in E \setminus \operatorname{int} V_1$, then

$$\begin{aligned} d(x,y) &= \inf\{r \in R^+;\ x - y \in K_r\} = \inf\{r \geq 1;\ x - y \in rV_1\} \\ &= \inf\{r \in R^+;\ r^{-1}(x-y) \in V_1\} = p_1(x-y)\,, \end{aligned}$$

where the last equality is true by definition of the Minkovski functional. If $x-y \in \operatorname{int} V_n \backslash 2 \operatorname{int} V_{n+1} = \operatorname{int} K_{2^{-n+1}} \backslash \operatorname{int} 2K_{2^{-n}}$, then obviously $d(x,y) \leq 2^{-n+1}$. Let us assume that $d(x,y) = s < 2^{-n+1}$, then $d(x,y) < s+\varepsilon$ for each $\varepsilon > 0$. When $s+\varepsilon < 2^{-n+1}$, this is means that $x-y \in \operatorname{int} K_{s+\varepsilon} = 2^n(s+\varepsilon) \operatorname{int} V_{n+1} \subset 2^n \cdot 2^{-n+1} \operatorname{int} V_{n+1} = 2 \operatorname{int} V_{n+1}$. But this is impossible, and therefore $d(x,y) = 2^{-n+1}$. Let it now $x-y \in \operatorname{int} 2V_{n+1} \backslash \operatorname{int} V_{n+1}$, then $d(x,y) = \inf\{r \in R^+;\ x-y \in K_r\} = \inf\{r \in [2^{-n}, 2^{-n+1}[;\ x-y \in K_r\}$. Indeed, $x-y \in 2 \operatorname{int} V_{n+1}$ and therefore $x-y \in (2-\varepsilon)V_{n+1}$ for some $\varepsilon > 0$, i.e. $x-y \in \frac{2^n(2-\varepsilon)}{2^n} V_{n+1} = 2^n s V_{n+1} = K_s$, where $s = \frac{2-\varepsilon}{2^n} < 2^{-n+1}$. Further, if $d(x,y) = l < 2^{-n}$, then, by virtue of the above, it is immediately obtained that $x-y \in \operatorname{int} V_{n+1}$. Hence,

$$d(x,y) = \inf\{r \in I_{n+1};\ x-y \in 2^n r V_{n+1}\} = 2^{-n} p_{n+1}(x-y)\,.$$

From the form (2.5.7) of the balls of the metric (2.5.8) it is obtained the following representation for Minkowski functionals

$$q_r(\cdot) = 2^{-n+1} r^{-1} p_n(\cdot),\ \text{for } r \in I_n\,. \tag{2.5.9}$$

If $(E, \|\cdot\|)$ is a normed space with unit ball S and $V_n = 2^{-n+1}S$, then the quasinorm $|\cdot|$ of the metric (2.5.8) defined by the equality $|x| = d(x,0)$, coincides with the original norm. $\square$

Corollary. *In the notation of Theorem* 2.5.7 *the following are true:*

a) *If* $|x| \in \operatorname{int} I_1$ (*respectively* $|x| \in \operatorname{int} I_n,\ n \geq 2$) *and* $\alpha \in \left[\frac{1}{p_1(x)}, \infty\right[$ (*respectively* $\alpha \in \left[\frac{1}{p_n(x)}, \frac{2}{p_n(x)}\right[$,) *then* $|\alpha x| = \alpha|x|$.

b) *If* $|x| = 2^{-n+1}$ $(n \in \mathbb{N})$ *and* $\alpha \in \left[\frac{2}{p_{n+1}(x)}, \frac{1}{p_n(x)}\right[$, *then* $|\alpha x| = |x|$.

Proof. A). Let $|x| \in \operatorname{int} I_1$ and $\alpha \in \left[\frac{1}{p_1(x)}, \infty\right[$, then $|x| = p_1(x)$ and $p_1(\alpha x) = \alpha p_1(x) = \alpha|x| \geq \frac{1}{p_1(x)} \cdot p_1(x) = 1$, i.e. $|\alpha x| = p_1(\alpha x) = \alpha p_1(x) = \alpha|x|$. Let now $|x| \in \operatorname{int} I_n =]2^{-n+1}, 2^{-n+2}[$ $(n \geq 2)$, then $|x| = 2^{-n+1}p_n(x)$ and $2^{-n+1} < 2^{-n+1}p_n(x) < 2^{-n+2}$, i.e. $1 < p_n(x) < 2$. Then $1 < p_n(\alpha x)$ and therefore $|\alpha x| = 2^{-n+1}p_n(\alpha x) = \alpha|x|$.

b). Let $|x| = 2^{-n+1}$, then $x \in V_n \backslash 2 \operatorname{int} V_{n+1}$. If $\alpha \in \left[\frac{2}{p_{n+1}(x)}, \frac{1}{p_n(x)}\right[$, then again $\alpha x \in V_n \backslash 2 \operatorname{int} V_{n+1}$, i.e. $|\alpha x| = 2^{-n+1} = |x|$.

If $(E, \mathfrak{T})$ is a metrizable LCS with a generating non-decreasing sequence of seminorms $\{\|\cdot\|_n\}$, then we put $p_n(x) = 2^{n-1}\|x\|_n$. Then the metric (2.5.8) takes view

$$d(x,y)=\begin{cases}\|x-y\|_1, & \text{when } x-y\in E\setminus \operatorname{int} V_1,\\ 2^{-n+1}, & \text{when } x-y\in \operatorname{int} V_n\setminus 2\operatorname{int} V_{n+1} (n\in\mathbb{N}),\\ \|x-y\|_{n+1}, & \text{when } x-y\in 2\operatorname{int} V_{n+1}\setminus \operatorname{int} V_{n+1}\\ & \qquad (n\in\mathbb{N}),\\ 0, & \text{when } x-y=0,\end{cases} \tag{2.5.10}$$

where $V_n=\{x\in E;\ p_n(x)\le 1\}$. And the Minkowski functionals of q_r balls K_r have the form

$$q_r(x)=r^{-1}\|\cdot\|_n, \quad \text{when} \quad r\in I_n\,. \tag{2.5.11}$$

This metric can be given a more convenient form

$$d(x,y)=\begin{cases}\|x-y\|_1, & \text{when } \|x-y\|_1\ge 1,\\ 2^{-n+1}, & \text{when } \|x-y\|_n\le 2^{-n+1}\\ & \qquad \text{and } \|x-y\|_{n+1}\ge 2^{-n+1},\\ \|x-y\|_{n+1}, & \text{when } 2^{-n}\le \|x-y\|_{n+1}<2^{-n+1}\\ & \qquad (n\in\mathbb{N}),\\ 0, & \text{when } x-y=0.\end{cases} \tag{2.5.12}$$

In spite of the fact that the balls of the metric K_r $(r\in R^+)$ are absolutely convex, the constructed metric is not norm-like. Indeed, due to corollaries of Theorem 2.5.7 quasinorm of the metric (2.5.8) is not strictly monotonic for points x with a quasinorm, which equal to 2^{-n+1} $(n\in\mathbb{N})$. In other words, the topological boundary ∂K_r of balls K_r coincides with the metric boundary $S_r=\{x\in E,\ |x|=r\}$ for $r\ne 2^{-n+1}$ $(n\in\mathbb{N})$. However, $\partial K_r\subset S_r$ and these sets, generally speaking, do not coincide for $r=2^{-n+1}$ $(n\in\mathbb{N})$. Let us give a simple example illustrating the mentioned situation. Let $C(R)$ be the space continuous real functions with the topology of compact convergence on R, which is given by the sequence of seminorms

$$p_n(x)=2^{n-1}\max\{|x(t)|;\ t\in[-n,n]\}, \quad n\in\mathbb{N},$$

and the basis of neighborhoods of zero $V_n=\{x\in E;\ p_n(x)\le 1\}$. Let's consider the functions

$$x_n(t)=\begin{cases}2^{-n+1}, & \text{when } t\in[-n,n],\\ 2^{-n+3}, & \text{when } t=\pm(n+1),\\ 0, & \text{when } t\in]-\infty,-(n+2)]\cup[n+2,\infty[\,,\\ \text{linear} & \text{in intervals } [-(n+2),-(n+1)],\\ & \quad [-(n+1),-n],[n,n+1] \text{ and } [n+1,n+2]\,.\end{cases}$$

Then we get that $x_n, x_n/2 \in V_n \backslash 2\,\mathrm{int}\,V_{n+1}$, i.e. $|x_n| = \left|\frac{x_n}{2}\right| = 2^{-n+1}$. To present the situation more clearly, let us indicate the form of the graph of the function $\alpha \to |\alpha x_1|$ on R^+.

By virtue of statement a) of the corollary of Theorem 2.5.7, for $\alpha \geq 1$ the equalities $|\alpha x_1| = \alpha|x_1| = \alpha$ are valid, and by virtue of statement b) of corollary of Theorem 2.5.7, for $\alpha \in \left[\frac{2}{p_2(x_1)}, \frac{1}{p_1(x_1)}\right] = \left[\frac{1}{4}, 1\right]$ the equalities are true $|\alpha x_1| = |x_1| = 1$. Next, consider the function $y = \frac{3}{16}x_1 \in 2V_2 \backslash \mathrm{int}\, V_2$. Therefore, $|y| = \frac{1}{2}p_2(y) = \frac{3}{4} \in \mathrm{int}\, I_2 = \left]\frac{1}{2}, 1\right[$. Again by virtue of statement a) of the corollary of Theorem 2.5.7, $|\beta y| = \beta|y| = \frac{3}{4}\beta$, when $\beta \in \left[\frac{2}{3}, \frac{4}{3}\right]$. Hence for $\alpha = \frac{16}{3}\beta \in \left[\frac{1}{8}, \frac{1}{4}\right]$ we get that $|\alpha x_1| = 4\alpha$. Let us now consider the function $y = \frac{1}{9}x_1$. Then $p_3(y) = \frac{16}{9}$, i.e. $y \in 2V_3 \backslash \mathrm{int}\, V_3$ and therefore $|y| = \frac{1}{4}p_3(y) = \frac{4}{9} \in \mathrm{int}\, I_3 = \left]\frac{1}{4}, \frac{1}{2}\right[$. This means that for $\beta \in \left[\frac{1}{p_3(y)}, \frac{2}{p_3(y)}\right] = \left[\frac{9}{16}, \frac{18}{16}\right]$ the equalities $|\beta y| = \beta|y| = \frac{4}{9}\beta$ are valid. From here for $\alpha = \frac{1}{3}\beta \in \left[\frac{1}{16}, \frac{1}{8}\right]$ we again obtain that $|\alpha x_1| = 4\alpha$. Considering the function $y = \frac{1}{17}x_1$ again we are convinced that $|\alpha x_1| = 4\alpha$ for $\alpha \in \left[\frac{1}{32}, \frac{1}{16}\right]$, etc. Due to the above, the graph of the function $\alpha \to |\alpha x_1|$ has the form of a non-decreasing broken line on $[0, \infty[$. □

Let us now consider the Fréchet space of all (equivalent classes) p-locally integrable on R functions $L^p_{loc}(R)$ $(1 \leq p < \infty)$. Basis of neighborhoods of the zero of the topology of this space is given by a sequence of neighborhoods

$$V_n = 2^{-n+1}B + \mathrm{Ker}\, p_n\,, \tag{2.5.13}$$

where

$$B = \left\{x \in L^p_{loc}(R);\ \left(\int_{-\infty}^{\infty} |x(t)|^p dt\right)^{1/p} \leq 1\right\}$$

and

$$p_n(x) = 2^{n-1}\left(\int_{-n}^{n} |x(t)|^p dt\right)^{1/p} \tag{2.5.14}$$

By virtue of (2.5.11), the following equalities are valid:

$$q_r = \begin{cases} r^{-1}\left(\int_{-1}^{1} |x(t)|^p dt\right)^{1/p}, & \text{when } r \in [1, \infty[\,, \\ r^{-1}\left(\int_{-n}^{n} |x(t)|^p dt\right)^{1/p}, & \text{when } r \in [2^{-n}, 2^{-n+1}[\ \ (n \in \mathbb{N}), \end{cases}$$

i.e. seminorms q_r $(r \in R^+)$ have the form L^p-norm. Moreover, R^+ is represented as the union of right half open right intervals I_n so that corresponding to the numbers r_1 and $r_2 \in I_n$ balls K_{r_1} and K_{r_2}, like to the balls of normed spaces, satisfy the equality $K_{r_1} = \frac{r_1}{r_2} K_{r_2}$. However, in non-normed spaces all balls cannot be similar each other. The constructed metric does not have this property only for the space $L^p_{loc}(R)$, but also for arbitrary metrizable LCS and this property will be multiple times be used in the future.

2.6 Representation of topologies of spaces of basic functions $D(\Omega)$ and generalized functions $D'(\Omega)$

In the previous sections, strict (LB)-spaces, which are discussed in relation to best approximation problems and the existence of interpolation splines, are studied. In this section, the representations of the topology of strict (LF)-spaces and their strong dual spaces are studied and used to represent the topologies of the spaces of basic functions $D(\Omega)$ and generalized functions $D'(\Omega)$. During the representation of the topologies of these spaces strict Fréchet–Hilbert spaces and their strongly dual spaces appeared for the first time.

2.6.1 Existence of a continuous metric on a strict (LF)-space

Definition of strict (LF)-spaces introduced by J. Dieudonne and L. Schwartz [43], is given in Chapter II, Section 2.2. When proving Proposition 2.2.2 we have extended the norm from the normed subspace of strict (LB)-space to the entire space. We do the same in case of strict (LF)-spaces when the metric is extended from its metrizable subspace on entire space.

Theorem 2.6.1. *Let $(E, \mathfrak{T})$ be the inductive limit of non-decreasing sequences of locally convex spaces $\{(E_n, \mathfrak{T}_n)\}$, i.e. $(E, \mathfrak{T}) = \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$. Then the following statements are equivalent:*

a) *$(E, \mathfrak{T})$ is a strict (LF)-space, i.e. $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$.*

b) *In $(E, \mathfrak{T})$ there is a non-increasing sequence of absolutely convex neighborhoods of zero $\{V_m\}$ such that $\cap_{m\in\mathbb{N}} V_m = \{0\}$ and $E_n \cap V_m = V_{nm}$ are bases of neighborhoods of zero of the topology $\mathfrak{T}_n$ for each $n \in \mathbb{N}$.*

c) *On $(E, \mathfrak{T})$ there is a continuous metric* (2.5.12) *inducing on each E_n topology $\mathfrak{T}_n$.*

Proof. a)⇒b). Let $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$ is strict (LF)-space and the topology $\mathfrak{T}_n$ of the Fréchet space E_n is generated by a non-decreasing sequence of seminorms $\{p_{nm}\}$. Let us denote by U_{nm} the sets

$$U_{nm} = \{x \in E,\ p_{nm}(x) \leq 1\}.$$

Obviously, for each $n \in \mathbb{N}$ the following inclusions are true: $U_{n1} \supset \ldots \supset U_{nm} \supset \cdots$. One can also assume that for each $n \in \mathbb{N}$ the following equalities hold: $\bigcap\limits_{m \in \mathbb{N}} U_{nm} = \{0\}$.

Since the imbedding $(E_1, \mathfrak{T}_1) \to (E_2, \mathfrak{T}_2)$ is topological monomorphism of the Fréchet space $(E_1, \mathfrak{T}_1)$ into the Fréchet space $(E_2, \mathfrak{T}_2)$, then for each $\mathfrak{T}_1$-neighborhood U_{1m} there is $\mathfrak{T}_2$-neighborhood U_{2k_m} such that $U_{2k_m} \cap E_1 \subset U_{1m}$. We can assume that for each $m \in \mathbb{N}$ the inclusion $U_{2k_m} \supset U_{2k_{m+1}}$ is true. The sequence $\{U_{2k_m}\}$ is also the basis of a neighborhood of zero in E_2 and denote it again through $\{U_{2m}\}$. Consider neighborhoods in E_2, defined by the relation $V_{2m} = \Gamma(U_{1m} \cup U_{2m})$, where the latter denotes absolutely convex hull of the set $U_{1m} \cup U_{2m}$. For each $m \in \mathbb{N}$ we have $V_{2,m+1} = \Gamma(U_{1,m+1} \cup U_{2,m+1}) \subset \Gamma(U_{1m} \cup U_{2m}) \subset V_{2m}$.

Obviously, for each $m \in \mathbb{N}$ the following equalities are true $V_{2m} \cap E_1 = U_{1m}$. Indeed, each $x \in V_{2m}$ has the following form: $x = \alpha x_1 + \beta x_2$, where $x_1 \in U_{1m}$, $x_2 \in U_{2m}$ and $|\alpha| + |\beta| \leq 1$. From the relation $\beta x_2 = x - \alpha x_1 \in E_1$ it follows that either $\beta = 0$, or $x_2 \in E_1$. In both cases $x \in U_{1m}$. Therefore, $U_{2m} \cap E_1 \subset U_{1m}$.

Let us now show that the sequence $\{V_{2m}\}$ forms a basis neighborhoods in E_2. Since for each $m \in \mathbb{N}$ it is true inclusion $U_{2m} \subset V_{2m}$, then it is enough to show that for each neighborhood U_{2k} there is a neighborhood V_{2k_l} such that $V_{2k_l} \subset U_{2k}$. Indeed, since $U_{2k} \cap E_1$-neighborhood in E_1, then there exists $\mathfrak{T}_1$-neighborhood U_{1k_l}, where $k_l \geq k$ such that $U_{2k} \cap E_1 \supset U_{1k_l}$. From here we have that

$$U_{1k_l} \cup U_{2k_l} \subset U_{2k} \cup U_{2k} = U_{2k} \,,$$

i.e.

$$V_{2k_l} \subset U_{2k} \,.$$

Let such basis of neighborhoods $\{V_{n-1,m}\}$ be already constructed in E_{n-1}. Repeating the reasoning given above, we can construct in E_n the basis of neighborhoods $\{V_{nm}\}$, satisfying the condition

$$V_{nm} \cap E_{n-1} = V_{n-1,m}$$

for each $n \in \mathbb{N}$. On the other hand, the following inclusions hold:

$$U_{1m} \subset V_{2m} \subset \cdots \subset V_{nm} \subset \cdots \,.$$

Therefore, the sets $V_m = \cup_{n \in \mathbb{N}} V_{nm}$, where $V_{1m} = U_{1m}$, are absolutely convex $\mathfrak{T}$-neighborhoods of zero in E. The sequence $\{V_m\}$ satisfies the conditions

$$V_m \bigcap E_n = V_{nm}$$

and

$$\bigcap_{m\in\mathbb{N}} V_m = \bigcap_{m\in\mathbb{N}} \Big(\bigcup_{n\in\mathbb{N}} V_{nm} \Big) = \bigcup_{n\in\mathbb{N}} \Big(\bigcap_{m\in\mathbb{N}} V_{nm} \Big) = \bigcup \{0\} = \{0\} .$$

The second equality takes place due to the ordering of the neighborhoods V_{nm} for fixed $n \in \mathbb{N}$.

b)⇒c). The sequence of absolutely convex neighborhoods $\{V_m\}$ generates a metrizable locally convex topology $\mathfrak{T}_{\{V_m\}}$ on E, which is not stronger than $\mathfrak{T}$. Let p_m be the Minkowski functional, corresponding to the neighborhood V_m. It is well known that the topology $\mathfrak{T}_{\{V_m\}}$ can be given using one of the metrics (2.5.2), (2.5.4) or (2.5.8). It is also obvious that this metric induces on each E_n the topology $\mathfrak{T}_n$.

c)⇒a). It immediately follows from the condition that for each $n \in \mathbb{N}$ the topology $\mathfrak{T}_{n+1}$ of the space E_{n+1}, induces a topology $\mathfrak{T}_n$ on E_n. □

Corollary 1. *Let $(E, \mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$ be a strict (LF)-space and the sequence $\{E_n\}$ is strictly increasing. Then the space $(E, \mathfrak{T}_{\{V_m\}})$ is not complete. However, $(E_n, \mathfrak{T}_n)$ are Fréchet subspaces of the space $(E, \mathfrak{T}_{\{V_m\}})$.*

Indeed, by virtue of Proposition 5 from ([144], p. 188) we obtain, that the space $(E, \mathfrak{T})$ is not metrizable. If the space $(E, \mathfrak{T}_{\{V_m\}})$ was complete, then by Theorem 1 ([43], p. 66) the identity mapping of the space $(E, \mathfrak{T})$ onto $(E, \mathfrak{T}_{\{V_m\}})$ would be a topological isomorphism. And this contradicts non-metrizability of the space $(E, \mathfrak{T})$. Completeness of subspaces E_n in $(E, \mathfrak{T}_{\{V_m\}})$ follows from the completeness of the space $(E, \mathfrak{T})$ and statement 4 ([144], p. 188).

Corollary 2. *Let $(E, \mathfrak{T}) = \lim(E_n, \mathfrak{T}_n)$ be the inductive limit of locally convex spaces $\{(E_n, \mathfrak{T}_n)\}$. Then the following statements are equivalent:*

a) *$(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$, where $(E_n, \mathfrak{T}_n)$ is metrizable locally convex spaces.*

b) *In $(E, \mathfrak{T})$ there is a sequence of absolutely convex neighborhoods of $\{V_m\}$ such that $\cap_{m\in\mathbb{N}} V_m = \{0\}$ and $E_n \cap V_m = V_{nm}$ are bases of neighborhoods of zero of topologies $\mathfrak{T}_n$ for each $n \in \mathbb{N}$.*

c) *On $(E, \mathfrak{T})$ there is a continuous metric* (2.5.12) *inducing on each E_n the topology $\mathfrak{T}_n$.*

Corollary 3. *Let $(E, \mathfrak{T})$ be a strict (LF)-space. In his strong dual space E' there is an increasing subsequence equicontinuous sets such that the $\sigma(E', E)$-closure their union coincides with E'.*

Indeed, if $\{V_m\}$ is a sequence of $\mathfrak{T}$-neighborhoods, satisfying the conditions of statement b) of Theorem 2.6.1, then it is easy to see that for the polars of these sets the statement of Corollary 3 is true. It should also be noted that, however, strong dual space does not satisfy the second countability axiom Mackey, i.e. there is no fundamental sequence in it of bounded absolutely convex sets. Locally convex spaces, possessing a sequence of equicontinuous sets, weak closure of the union of which coincides with E', were studied in [177].

Existence of a continuous metric on an LCS is an important fact for applications. For example, from existence of a metric on a strict (LF)-space using Theorem 3 of J. Dieudonne and L. Schwartz ([82], p. 311) it is immediately obtained relatively weak sequential compactness relatively weakly countably compact sets ([82], p. 312). In [51], it was proved that a continuous metric exists on LCS $(E,\mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n,\mathfrak{T}_n)$, where $(E_n,\mathfrak{T}_n)$ is locally convex spaces on which there exist continuous metrics. This theorem was proved using step-by-step continuation of metrics from the first subspace to the second, etc. Interesting results on the continuation of the norm from a subspace to the entire space were obtained in [37, 51].

2.6.2 Representations of the topology of strict (LF)-spaces and its strong dual spaces. Applications to the spaces D and D'

Let $(E,\mathfrak{T})$ be a strict (LF)-space and let $\mathcal{V}$ be a basis of zero's neighborhoods of the topology $\mathfrak{T}$. In what follows, we denote by $\mathfrak{T}_V$, $V \in \mathcal{V}$, metrizable locally convex topology $\mathfrak{T}_{\{V_m\}}$, satisfying conditions of statement c) of Theorem 2.6.1, and for which $V = V_1$.

Proposition 2.6.2. *Strict (LF)-space $(E,\mathfrak{T})$ is topologically isomorphic to the locally convex kernel of all metric spaces $\{(E,\mathfrak{T}_V);\ V \in \mathcal{V}\}$, with respect to identity mappings $E\to(E,\mathfrak{T}_V)$, i.e. topology $\mathfrak{T}$ coincides with the upper bound of the topology of family $\{\mathfrak{T}_V; V \in \mathcal{V}\}$.*

Proof. Let $\mathfrak{T}'$ be a topology of the above-mentioned locally convex kernel. Since the identity mappings $(E,\mathfrak{T}) \to (E,\mathfrak{T}_V)$ are continuous and $\mathfrak{T}'$ is the weakest such topology, then $\mathfrak{T}' \le \mathfrak{T}$. If now $V \in \mathcal{V}$ and $\{V_m\}$ are a sequence of $\mathfrak{T}$-neighborhoods, satisfying the conditions of statement b) of Theorem 2.6.1, then the sequence $V, V\cap V_1, \ldots, V\cap V_m, \ldots$ again satisfies the specified conditions. Due to the continuity of the identity map $(E,\mathfrak{T}) \to (E,\mathfrak{T}_V)$ we immediately obtain that V is $\mathfrak{T}'$-neighborhood, i.e. $\mathfrak{T} \le \mathfrak{T}'$. This means $(E,\mathfrak{T}) = \mathbb{K}_{V\in\mathcal{V}}(E,\mathfrak{T}_V)$. From the proof it is clear that the family metrizable topologies $\{\mathfrak{T}_V;\ V \in \mathcal{V}\}$ can be chosen so that its cardinality was equal to the cardinality of the given basis of the zero's neighborhoods $\mathcal{V}$. $\square$

Corollary 1. *Let $(E,\mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n,\mathfrak{T}_n)$ be a strict (LF)-space, and $B\subset E$ is a subset of the space E. Then the following statements are equivalent:*

a) *B is bounded $($precompact$)$.*

b) *B is contained and bounded $($precompact$)$ in some Fréchet space E_n.*

c) *B is bounded $($precompact$)$ in every metrizable space $(E,\mathfrak{T}_V)$, where $V\in\mathcal{V}$.*

d) *B is contained in some E_n and is bounded $($precompact$)$ in some $(E,\mathfrak{T}_V)$.*

Proof. Equivalence a)⇔b) is proven in ([144], p. 188). a)⇔c) follows from the Proposition 2.6.2 by virtue of Theorem 7 ([82], p. 227). a)⇒d) follows from statements b) and c). d)⇒a) is true due to the properties of the topology $\mathfrak{T}_V$. □

Corollary 2. *Let $(E,\mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n,\mathfrak{T}_n)$ be a strict (LF)-space. Then the following statements are equivalent:*

a) *The sequence $\{x_k\}$ converges to x in $(E,\mathfrak{T})$.*

b) *For each $k\in\mathbb{N}$ x_k, $x\in E_{n_0}$ for some $n_0\in\mathbb{N}$ and $\{x_k\}$ converges to x in $(E_{n_0},\mathfrak{T}_{n_0})$.*

c) *The sequence $\{x_k\}$ converges to x in every space $(E,\mathfrak{T}_V)$, where $V\in\mathcal{V}$.*

d) *For each $k\in\mathbb{N}$ x_k, $x\in E_{n_0}$, for some $n_0\in\mathbb{N}$ and $\{x_k\}$ converges to x in some space $(E,\mathfrak{T}_V)$.*

It should be noted that convergence in the strict (LF)-space $(E,\mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n,\mathfrak{T}_n)$ is a simple quasimetric convergence $c = c(d_V,f)$ ([48], p. 492), where d_V is a metric on E, generating the topology $\mathfrak{T}_V$, and f is a function on E, defined by the equality:

$$f(x) = \min\{n;\ x\in E_n\}\,.$$

It should also be noted that precisely according to statement b) of Corollary 2 defines convergence in many known strict (LF)-spaces.

Let $(E,\mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(E_n,\mathfrak{T}_n)$ be a strict (LF)-space with a basis of neighborhoods of zero $\mathcal{V}$ and $\{\mathfrak{T}_\alpha;\ \alpha\in A\}$ be a family of all metrizable topologies $\mathfrak{T}_\alpha,\ \alpha\in A$ on E, inducing on each E_n topology $\mathfrak{T}_n$.

This family can be made directed by setting $\mathfrak{T}_\alpha = \mathfrak{T}_{\{V_m\}} \le \mathfrak{T}_{\{V'_m\}} = \mathfrak{T}_\beta$, if for any neighborhood V_m there exists a neighborhood V'_{k_m} and the number $\lambda_m > 0$ such that $V'_{k_m}\subset\lambda_m V_m$, i.e. if the topology $\mathfrak{T}_{\{V_m\}}$ is not stronger than the topology $\mathfrak{T}_{\{V'_m\}}$. For such topologies, identity mappings $\pi_{\alpha\beta} : (E,\mathfrak{T}_\beta)\to(E,\mathfrak{T}_\alpha)$ are continuous. Let $\overline{\pi}_{\alpha\beta}$ be a continuous extension on $(\widetilde{E_\beta,\mathfrak{T}_\beta})$ of mapping $\pi_{\alpha\beta}$, where $(\widetilde{E_\beta,\mathfrak{T}_\beta})$ is completion of the space $(E,\mathfrak{T}_\beta)$.

Theorem 2.6.3. *Let* $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$ *be strict* (LF)*-space and* $\{\mathfrak{T}_\alpha;\ \alpha \in A\}$ *be the family of all metrizable locally convex topologies* $\mathfrak{T}_\alpha$ *on* E*, inducing on each* E_n *topology* $\mathfrak{T}_n$*. Then the following statements are valid:*

a) *The space* $(E, \mathfrak{T})$ *is topologically isomorphic to the projective limit of the family of Fréchet spaces* $\{\widetilde{(E_\alpha, \mathfrak{T}_\alpha)};\ \alpha \in A\}$ *with respect to mappings* $\overline{\pi}_{\alpha\beta}$.

b) *If the space* $(E, \mathfrak{T})$ *is nuclear* (*respectively of type* (S)*, i.e. Schwartz space*)*, then Fréchet spaces* $\widetilde{(E_\alpha, \mathfrak{T}_\alpha)}$ *one can choose nuclear* (*respectively of type* (FS)*, i.e. as Fréchet-Schwartz*) *spaces.*

Proof. a) Consider the projective limit $F = \lim\limits_{\leftarrow}(E, \mathfrak{T}_\alpha)$. The topology of the space F is the induced topology from the product $\prod_{\alpha\in A}(E, \mathfrak{T}_\alpha)$. The mapping from E to F is linear and injective. From Corollary 2 of Proposition 2.6.2 it follows that it is sequentially continuous, and due to the bornologicality of the space $(E, \mathfrak{T})$, is continuous. In addition, since $(E, \mathfrak{T}) = K_{\alpha\in A}(E, \mathfrak{T}_\alpha)$ then, according to Theorem 1 ([82], p. 230), $(E, \mathfrak{T})$ is topologically isomorphic subspace E^0 of the projective limit F. From the above we find that $E^0 = F$, i.e. $(E, \mathfrak{T})$ is topologically isomorphic to the space F. On the other hand, one can prove that F is a dense subspace of the projective limit $\lim\limits_{\leftarrow}\widetilde{(E, \mathfrak{T}_\alpha)}$ with respect to the mappings $\overline{\pi}_{\alpha\beta}$. But F, being isomorphic to E, is complete and therefore $(E, \mathfrak{T}) = \lim\limits_{\leftarrow}\widetilde{(E, \mathfrak{T}_\alpha)}$.

b) Let $(E, \mathfrak{T})$ be nuclear, V be $\mathfrak{T}$-neighborhood and $\{V_m\}$ be a sequence of $\mathfrak{T}$-neighborhoods satisfying the conditions of statement b) of Theorem 2.6.1, and $V = V_1$. Let p_{V_1} be the Minkowski functional for V_1. By E_{V_1} we denote the space $E/Kerp_{V_1}$ with the associated norm $\widehat{p}_{V_1}$. Due to nuclearity $(E, \mathfrak{T})$ for V_1 exists the $\mathfrak{T}$-neighborhood U such that the canonical map of the normed spaces E_U on E_{V_1} is nuclear. Therefore the canonical mapping of $E_{U\cap V_2}$ onto E_{V_1} is nuclear, as the product of a continuous mapping $E_{U\cap V_2}$ to E_U and nuclear mapping E_U to E_{V_1}. Now denoting $U \cap V_2$ again by V_2 and repeating the reasoning given above, we will construct a sequence $\{V_m\}$, satisfying again the conditions of statement b) of Theorem 2.6.1 and such that the canonical mapping $E_{V_{m+1}}$ to E_{V_m} is nuclear for every $m \in \mathbb{N}$. Consequently, if the strict (LF)-space $(E, \mathfrak{T})$ is nuclear, then metrizable spaces $(E, \mathfrak{T}_\alpha),\ \alpha \in A$ and their completions can be chosen nuclear.

If the space $(E, \mathfrak{T})$ is a space of type (S), then analogously to the above, one can prove the completely boundedness of the mapping $E_{V_{m+1}}$ onto E_{V_m} for each $m \in \mathbb{N}$. Consequently, in this case, the space $\widetilde{(E, \mathfrak{T}_\alpha)},\ \alpha \in A$ can be chosen as Fréchet-Schwarz spaces (of type (FS)). $\square$

Corollary. *Let* $(E, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(E_n, \mathfrak{T}_n)$ *be a strict* (LF)*-space. In the notation of Theorem* 2.6.3*, for the dual space* E'*, the following statements are valid:*

a) *The space* $E' = \cup_{\alpha \in A} E'_\alpha$ *and* E'_α *is weakly dense in* E' *for each* $\alpha \in A$. *Further, if the space* $(E, \mathfrak{T})$ *is nuclear* (*resp. of type* (S)), *then the space* $(E', \beta(E', E))$ *is isomorphic to the inductive limit of the family of nuclear* (LB)-*spaces (resp. spaces of type* (DFS)) $\{(E'_\alpha, \beta(E'_\alpha, E_\alpha));\ \alpha \in A\}$ *with respect to mappings* $\overline{\pi}'_{\alpha\beta}$.

b) *If the spaces* $(E_n, \mathfrak{T}_n)$, $n \in \mathbb{N}$, *are distinguished, then for any* $n \in \mathbb{N}$ *and* $\alpha \in A$ *the spaces* $(E'_n, \beta(E'_n, E_n))$, $(E'/E_n^\perp, \beta(E', E)/E_n^\perp)$ *and* $(E'_\alpha/E_{n,\alpha}^\perp, \beta(E'_\alpha, E_\alpha)/E_{n,\alpha}^\perp)$ *are isomorphic, where* $E_n^\perp$ *and* $E_{n,\alpha}^\perp$ *are orthogonal* (*in the sense of dual pairs* $\langle E, E' \rangle$ *and* $\langle E_\alpha, E'_\alpha \rangle$, *respectively*) *to the subspaces* E_n *of the spaces* E' *and* E'_α, *respectively. Further, for each* $\alpha \in A$, *the space* $(E', \beta(E', E))$ *is isomorphic to the projective limit of a sequence of* (LB)-*spaces* $\{(E'_\alpha/E_{n,\alpha}^\perp, \beta(E'_\alpha, E_\alpha)/E_{n,\alpha}^\perp)\}$ *with respect to topological homomorphisms* j'_n, *where* j_n *is topological monomorphism of the space* $(E_n, \mathfrak{T}_n)$ *in* $(E_{n+1}, \mathfrak{T}_{n+1})$.

c) *If the space* $(E_n, \mathfrak{T}_n)$ *is reflexive* (*nuclear, type* (S)), *then the spaces* $(E'_\alpha/E_{n,\alpha}^\perp, \beta(E'_\alpha, E_\alpha)/E_{n,\alpha}^\perp)$ *are reflective* (LB)-*spaces* (*nuclear* (LB)-*spaces, spaces type* (DFS)).

Proof. a) The first equality follows from the fact that a linear functional on a locally convex space $(E, \mathfrak{T})$ is continuous if and only if it is bounded in some neighborhood of zero. Since identical the mapping $(E, \mathfrak{T})$ onto $(E, \mathfrak{T}_\alpha)$ is injective, then $(E, \mathfrak{T}_\alpha)'$ is weakly dense in E'.

Let $(E, \mathfrak{T})$ be a nuclear strict (LF)-space. Because metrizable locally convex spaces $E_\alpha = (E, \mathfrak{T}_{\{V_m\}})$ in such case, it is possible choose nuclear ones, then $(E'_\alpha, \beta(E'_\alpha, E_\alpha)) = \lim\limits_{\rightarrow} E'_{V_m^0}$, where $E'_{V_m^0}$ is Banach space spanned by V_m^0. On the other hand, by virtue of nuclearity, the space $(E, \mathfrak{T})$ has the property (S) ([147], p. 432), i.e. is represented as a projective limit with completely continuous mappings. Therefore, $(E', \beta(E', E)) = \lim\limits_{\rightarrow} E'_{V^0}$, where V runs through the basis of neighborhoods of zero $\mathcal{V} = U_0(E)$. By virtue of transitivity of the formation of a locally convex hulls ([82], p. 217), we also have that $(E', \beta(E', E))$ is a locally convex hull of the spaces $(E'_\alpha, \beta(E'_\alpha, E_\alpha))$ relative to imbeddings $(E'_\alpha, \beta(E'_\alpha, E_\alpha)) \to E'$. Since conjugate to the identity map $\pi_{\alpha\beta} : (E, \mathfrak{T}_\beta) \to (E, \mathfrak{T}_\alpha)$ is continuous in strong topologies, then $(E', \beta(E', E)) = \lim\limits_{\rightarrow}(E'_\alpha, \beta(E'_\alpha, E_\alpha))$. By similar reasoning one can prove this statement in case of a space $(E, \mathfrak{T})$ of type (S).

b) By ([65], Theorem 8) the strong dual to the distinguished subspace $(E_n, \mathfrak{T}_n)$ is identified with quotient spaces

$$(E'/E_n^\perp, \beta(E', E)/E_n^\perp) \quad \text{and} \quad (E'_\alpha/E_{n,\alpha}^\perp, \beta(E'_\alpha, E_\alpha)/E_{n,\alpha}^\perp),$$

where $E_n^\perp$ and $E_{n,\alpha}^\perp$ are subspaces of the spaces E' and E'_α, orthogonal to E_n, respectively. From the distinguishness of subspaces $(E_n, \mathfrak{T}_n)$, we obtain also that the conjugate to the monomorphism $j_n : (E_n, \mathfrak{T}_n) \to (E_{n+1}, \mathfrak{T}_{n+1})$ is a strong homomorphism. The rest of our statement follows from the fact that the strong dual to the inductive limit sequences of locally convex spaces is projective limit of strong dual spaces with respect to conjugate mappings ([82], p. 290).

c) If the space $(E_n, \mathfrak{T}_n)$ is reflexive (nuclear, type (S)), then the spaces $(E'_\alpha/E^\perp_{n,\alpha}, \beta(E'_\alpha, E_\alpha)/E^\perp_{n,\alpha})$ are reflective (LB)-spaces (nuclear (LB)-spaces, spaces type (DFS)). The rest of statement c) follows from statement b).

We now assume that on a strict (LF)-space $(E, \mathfrak{T}) = s \cdot \lim\limits_{\to}(E_n, \mathfrak{T}_n)$ there is a basis of zero's neighborhoods $\mathcal{V}$ such that the Minkowski functional p_V for $V \in \mathcal{V}$ is norm on E. Let $p_{V,n}$ be the restriction of p_V to E_n, $(E_V, \mathfrak{T}_V) = s \cdot \lim\limits_{\to}(E_n, p_{V,n})$ is strict inductive limit of a sequence of normed spaces $\{(E_n, p_{V,n})\}$, $\mathfrak{T}_V$ is its topology and $(\widetilde{E_V, \mathfrak{T}_V}) = s \cdot \lim\limits_{\to}(\widetilde{E_n, p_{V,n}})$ is strict (LB)-space, where $(\widetilde{E_n, p_{V,n}})$ is the completion of the normed space $(E_n, p_{V,n})$. In these conditions it is valid

Theorem 2.6.4. *Let $(E, \mathfrak{T}) = s \cdot \lim\limits_{\to}(E_n, \mathfrak{T}_n)$ be a strict (LF)-space with a basis of zero's neighborhoods $\mathcal{V}$. Then the following statements are valid:*

a) *$(E, \mathfrak{T})$ is isomorphic to the projective limit of the family of strict (LB)-spaces $\{(\widetilde{E_V, \mathfrak{T}_V});\ V \in \mathcal{V}\}$ with respect to mappings $\overline{g}_{UV}$, where $\overline{g}_{UV}$ is a continuous extension to $(\widetilde{E_V, \mathfrak{T}_V})$ of identity mapping $(E_V, \mathfrak{T}_V)$ on $(E_U, \mathfrak{T}_U)$ $(V \subset U,\ U, V \in \mathcal{V})$. In addition, $E = \cap_{V \in \mathcal{V}} E_V$, $\mathfrak{T}$ coincides with the supremum of the family $\{\mathfrak{T}_V;\ V \in \mathcal{V}\}$ in the topology lattice on E. The dual space $E' = \cup_{V \in \mathcal{V}} E'_V$ and E'_V are weakly dense in E' for each $V \in \mathcal{V}$.*

b) *For each $V \in \mathcal{V}$, the space $(E'_V, \beta(E'_V, E_V))$ is quojection and $(E''_V, \beta(E''_V, E'_V))$ is a strict (LB)-space. Next, if the spaces $(E_n, \mathfrak{T}_n)$ are distinguished, then the space $(E', \beta(E', E))$ is isomorphic to the inductive limit of the family $\{(E'_V, \beta(E'_V, E_V));\ V \in \mathcal{V}\}$ with respect to conjugate maps $\overline{g}'_{UV}$.*

c) *If the space $(E, \mathfrak{T})$ is nuclear (i.e., the spaces $(E_n, \mathfrak{T}_n)$ are nuclear), then the spaces $(E_V, \mathfrak{T}_V)$ can be chosen as strict (LH)-spaces, and therefore the strong dual spaces $(E'_V, \beta(E'_V, E_V))$; $V \in \mathcal{V}$ can be chosen as strict Fréchet–Hilbert spaces.*

Proof. a) Let $I_V : E \to (E_V, \mathfrak{T}_V)$ be the identical mapping. The topology $\mathfrak{T}$ is a projective topology with respect to families $\{(E_V, \mathfrak{T}_V, I_V);\ V \in \mathcal{V}\}$, i.e. weakest topology on E, for which all mappings I_V, $V \in \mathcal{V}$ are continuous ([147], p. 68). Further, $\mathfrak{T}$ is the supremum topology of the family $\{\mathfrak{T}_V;\ V \in \mathcal{V}\}$. Moreover, the space $(E, \mathfrak{T})$ is projective limit of the spaces $\{(\widetilde{E_V, \mathfrak{T}_V});\ V \in \mathcal{V}\}$ with respect to mappings $\overline{g}_{UV}$ ([147], p. 70). The completeness of strict (LF)-spaces implies equality $E = \cap_{V \in \mathcal{V}} \widetilde{E}_V$. It is obvious that $E' = \cup_{V \in \mathcal{V}} E'_V$. The conjugate mapping

I'_V to the identity mapping I_V is injective and has weakly dense image. Therefore, the space E'_V is weakly dense in E' for each $V \in \mathcal{V}$.

b) The first part of this statement follows from the theorem 2.3.2. Due to the distinguishness of Fréchet spaces $(E_n, \mathfrak{T}_n)$ and Theorem 10 from [65] it follows that the space $(E', \beta(E', E))$ is bornological. Therefore $(E', \beta(E', E))$ is the inductive limit of the family of Banach spaces E_B, where B is bounded, closed and absolutely convex subset of the space $(E', \beta(E', E))$, and E_B is space spanned by B. On the other side, for each $V \in \mathcal{V}$ the space $(E'_V, \beta(E'_V, E_V))$ is also the inductive limit of corresponding spaces E_B. Therefore, due to transitivity formation of the inductive limit, the space $(E', \beta(E', E))$ is isomorphic to the inductive limit of the family of quojections $\{(E'_V, \beta(E'_V, E_V)); V \in \mathcal{V}\}$.

c) If the space $(E, \mathfrak{T})$ is nuclear, then, as is well known, the neighborhood basis $\mathcal{V}$ can be chosen so that for each $V \in \mathcal{V}$ Minkowski functional p_V is generated by the inner product. Then the spaces $(\widetilde{E_V, \mathfrak{T}_V})$ will be strict (LH)-spaces. Next, it remains to apply sentemce b). $\square$

It should be noted that, in contrast to the representation of nuclear (LF)-spaces and their strong dual spaces in the form of projective and inductive limits of the family of Hilbert spaces [125], the strict (LF)-space $(E, \mathfrak{T})$ is not represented in the form of inductive and projective limits of families of strict Fréchet–Hilbert spaces, and its strong dual space $(E', \beta(E', E))$ is not represented in the form of projective and inductive limits families of strict (LH)-spaces.

Example. Space of basic (finite) functions $D(\Omega)$. Let Ω be a non-empty open set in R^l and $\{\Omega_n\}$ be an increasing sequence of compact sets in Ω such that $\Omega_n \subset int\,\Omega_{n+1}$ for each $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$. As is well known, through $D(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. The space $D(\Omega)$ is the strict inductive limit of the increasing sequence of nuclear countable-normed spaces $D_n = C_0^\infty(\Omega_n)$, i.e. spaces of functions from D equal to zero outside the compact set Ω_n. Indeed, each D_n is endowed with topology by a sequence of norms

$$p_{\Omega_{n,m}}(f) = \sup\{|D^{(S)}f(t)|;\ t \in \Omega_n,\ |s| \le m\}, \quad n, m \in \mathbb{N}\,, \tag{2.6.1}$$

where

$$s = (s_1 \ldots, s_l), \quad |s| = \sum_{i=1}^{l} s_i, \quad D^{(s)}x(t) = \frac{\partial^{s_1+\cdots+s_l} f(t_1, \ldots, t_l)}{\partial t_1^{s_1} \cdots \partial t_l^{s_l}}\,.$$

Then D_n turns into a Fréchet space, and its topology coincides with the induced topology from the space D_{n+1} on D_n, i.e. $D(\Omega) = s \cdot \lim\limits_{\rightarrow}(D_n, \mathfrak{T}_n)$.

By virtue of Theorem 2.6.1 on the space $D(\Omega)$ there exist metrizable topologies inducing on each D_n the topology of the space D_n. As such topologies we present the topologies defined with a countable sequence of norms on $D(\Omega)$

$$p_m(f) = \sup\{|D^{(S)} f(t)|;\ t \in \Omega_m,\ |s| \leq m\},\ m \in \mathbb{N}\,, \tag{2.6.2}$$

$$p_{mr}(f) = \sup\left\{(1+|t|^r) \sum_{|S| \leq m} |D^{(s)} f(t)|;\ t \in \Omega_m\right\}, \quad r, m \in \mathbb{N}\,, \tag{2.6.3}$$

$$\|f\|_m = \sup\left\{ \sum_{|S| \leq m} |D^{(s)} f(t)|;\ t \in \Omega\right\}, \quad m \in \mathbb{N}\,. \tag{2.6.4}$$

From Corollary 1 of Theorem 2.6.1 it follows that in the case $\Omega = R^l$ the space $D(R^l)$, considered by these metrizables topologies,is not complete. By completing the space $D(R^l)$ by topology with the sequence (2.6.1) or (2.6.2) it is obtained the universal nuclear space of all infinite differentiable functions $\mathcal{E}(R^l)$ with the topology of compact convergence, i.e. with the topology of uniform convergence of derivatives of all orders on compact sets from R^l. By completing the space $D(R^l)$ with the topology of a sequence of norms (2.6.3) a nuclear countable-normed Schwartz space $S(R^l)$ is obtained. In [194], other examples of metrizable locally convex spaces inducing on each D_n topology $\mathfrak{T}_n$ are considered.

A continuous linear functional defined on an LCS $(D, \mathfrak{T})$, is called a distribution or generalized function of L. Schwartz. Therefore, for every generalized function $F' \in D'(\Omega)$ there is a $\mathfrak{T}$-neighborhood V, for which $F' \in V^0$. Therefore, by virtue of the corollary of Theorem 2.6.3, F' is also continuous linear functional on some nuclear metrizable space D_α, inducing on each subspace $(D_n, \mathfrak{T}_n)$ original topology. For example, generalized functions with compact support is the nuclear (LB)-space $\mathcal{E}'(R^l)$. The space of tempered distributions on R^l is the nuclear (LB)-space $S'(R^l)$. More precisely, from the above results about strict (LF)-spaces, for the space $(D, \mathfrak{T})$ it follows that the following is true

Theorem 2.6.5. *Let the space* $(D, \mathfrak{T}) = s\cdot\lim\limits_{\rightarrow}(D_n, \mathfrak{T}_n)$ *and* $\{\mathfrak{T}_\alpha;\ \alpha \in A\}$ *be a family of all metrizable locally convex topologies on* D*, inducing on each* D_n *the topology* $\mathfrak{T}_n$*. Then the following statements are true:*

a) *The space* D *is isomorphic to the projective limit of the family of nuclear Fréchet spaces* $\{\widetilde{(D_\alpha, \mathfrak{T}_\alpha)};\ \alpha \in A\}$ *with respect to the mappings* $\overline{\pi}_{\alpha\beta}$*, where* $\widetilde{D}_\alpha = \widetilde{(D_\alpha, \mathfrak{T}_\alpha)}$ *is the completion of spaces* $(D, \mathfrak{T}_\alpha)$*, and* $\overline{\pi}_{\alpha\beta}$ *is continuous extension to* $\widetilde{D}_\beta$ *of the identical mappings from* $(D, \mathfrak{T}_\beta)$ *to* $(D, \mathfrak{T}_\alpha)$ $(\mathfrak{T}_\alpha \leq \mathfrak{T}_\beta)$ *and the topology of the space* D *coincides with the supremum of the family of topologies* $\{\mathfrak{T}_\alpha;\ \alpha \in A\}$*. The space* $D' = \cup_{\alpha \in A} D'_\alpha$ *and* D'_α *is weakly dense in* D' *for each* $\alpha \in A$*.*

b) *The space D' in the strong topology $\beta(D', D)$ is isomorphic to the projective limit of the family of nuclear (LB)-spaces $\{(D'_\alpha, \beta(D'_\alpha, D_\alpha)),\ \alpha \in A\}$ with respect to mappings $\overline{\pi}'_{\alpha\beta}$.*

c) *For each $\alpha \in A$ the space $(D', \beta(D', D))$ is isomorphic to the projective limit of a sequence of nuclear (LB)-spaces $(D'_\alpha / D^{\perp}_{n,\alpha}, \beta(D'_\alpha, D_\alpha)/D^{\perp}_{n,\alpha})$ with respect to topological homomorphisms j'_n, where $D^{\perp}_{n,\alpha}$ is the subspace of D'_α, orthogonal to D_n and j_n is monomorphism of the space $(D_n, \mathfrak{T}_n)$ into $(D_{n+1}, \mathfrak{T}_{n+1})$.*

d) *If $\mathcal{V}$ is a basis of neighborhoods of zero in D, then the space $\widetilde{(D, \mathfrak{T})}$ is isomorphic to the projective limit of the family of strict $\widetilde{(LB)}$-spaces $\{(D_V, \mathfrak{T}_V);\ V \in \mathcal{V}\}$ with respect to mappings $\overline{g}_{UV}$, where $\widetilde{D}_V = \widetilde{(D_V, \mathfrak{T}_V)}$ is completion of space $(D, \mathfrak{T}_V) = s \cdot \lim\limits_{\rightarrow}(D_n, p_{V,n})$, and $\overline{g}_{UV}$ is continuous extension on $\widetilde{D}_V$ of the identity map $(D, \mathfrak{T}_V)$ on $(D, \mathfrak{T}_U)$ $(\mathfrak{T}_U \leq \mathfrak{T}_V)$. $D = \cap_{V \in \mathcal{V}} D_V$, the topology of the space D coincides with the supremume of the family $\{\mathfrak{T}_V;\ V \in \mathcal{V}\}$, $D' = \cup_{V \in \mathcal{V}} D'_V$ and D'_V is weakly dense in D' for each $V \in \mathcal{V}$.*

e) *The space $(D', \beta(D', D))$ is isomorphic to the inductive limit of families of quojections $\{D'_V, \beta(D'_V, D_V))\}$ with respect to mappings $\overline{g}'_{UV}$.*

It should be noted that if the basis of neighborhoods of zero $\mathcal{V}$ of space D is chosen so that for each $V \in \mathcal{V}$ the Minkowski functional p_V is generated by the inner product, then in statement d), the spaces D_V can be chosen as strict (LH)-spaces, and in statement e), the spaces D'_V as strict Fréchet–Hilbert spaces for each $V \in \mathcal{V}$.

Let us now give examples of strict (LB)-spaces appearing in statement d) of Theorem 2.6.4. As is known (see Section 2.3), $K(\Omega)$ denotes the space of functions continuous on Ω with compact support. In the topology of the inductive limit $K(\Omega)$ is a strict (LB)-space, since $K(\Omega) = s \cdot \lim\limits_{\rightarrow} K(\Omega, \Omega_n)$, where $K(\Omega, \Omega_n)$ is the space of continuous functions on Ω, whose supports are contained in Ω_n. Also $\mathcal{L}^p_0(\Omega)$ $(1 < p < \infty)$ is reflexive (LB)-space of all finite in Ω functions from Banach space of p-summable functions $L^p(\Omega)$, since $\mathcal{L}^p_0(\Omega) = s \cdot \lim\limits_{\rightarrow} \mathcal{L}^p_0(\Omega_n)$ ($\mathcal{L}^2_0(\Omega_n)$ is strict (LH)-space), where $\mathcal{L}^p_0(\Omega_n)$ is the space of all functions of $L^p(\Omega)$, which vanish almost everywhere outside Ω_n.

According to statement e) of Theorem 2.6.4 the quojections that are strong dual to the specified strict (LB)-spaces, participate in the representation of space $(D', \beta(D', D))$. In particular, such is the space of measures $M(\Omega) = K(\Omega)'$ in the strong topology. Also, strong dual to space $\mathcal{L}^p_0(\Omega)$ is the space of q-locally summable functions $L^q_{loc}(\Omega)\big(\frac{1}{p} + \frac{1}{q} = 1\big)$. From these results in particular, it is obtained

Proposition 2.6.6. *In order for a generalized function $f \in D'(\Omega)$ was a measure*

(q-locally integrable function $(1 < q < \infty)$) on Ω, it is necessary and sufficient that it admits linear and continuous extension on $K(\Omega)$ $\left(\mathcal{L}_0^p(\Omega), \frac{1}{p} + \frac{1}{q} = 1\right)$.

It should be noted that the obtained representations of nuclear strict (LF)-spaces and its strong dual spaces are important in particular for nuclear spaces of basic functions $D(Q_p)$ and generalized functions $D'(Q_p)$, where Q_p is the field of p-adic numbers. In this case it is also possible to define the strict Fréchet–Hilbert space $L^2_{loc}(Q_p)$, as well as the families of strict Fréchet–Hilbert spaces and nuclear (LH)-spaces participating in representation of the topology $D(Q_p)$.

2.6.3 Sobolev space of infinite order and embedding theorems

Let again $\Omega \subset R^l$ be a non-empty open set. Sobolev spaces of finite order $W_p^n(\Omega)$ $(p \geq 1, n \in \mathbb{N})$ are important Banach spaces of generalized functions. We connect these Sobolev spaces to Sobolev spaces of infinite order and study the properties of embedding operators. We will say that $f \in W^{p,\infty}(\Omega)$, if f has generalized derivatives of all orders $f^{(\alpha)} \in L^p(\Omega)$ ($\alpha = (\alpha_1, \ldots, \alpha_l)$ is multi-index). It is natural to consider the space $W^{p,\infty}(\Omega)$ with a topology that coincides with $L^p(\Omega)$ convergence of derivatives of all orders. This topology is not normable, but is metrizable and can be given by a non-decreasing sequence of norms

$$\|f\|_{p,n} = \left(\sum_{|\alpha| \leq n} \|f^{(\alpha)}\|_p^p\right)^{1/p}, \quad |\alpha| = \sum_{i=1}^{l} \alpha_i, \quad n \in \mathbb{N},$$

where

$$\|f\|_p = \left(\int_\Omega |f(t)|^p dt\right)^{1/p}.$$

In [191], it was proved that the space $W^{p,\infty}(R^l)$ is complete countable-normed space. For an arbitrary open set Ω, the space $W^{p,\infty}(\Omega)$ is the countable Hilbert space. From the well-known representation of complete countable-normed spaces it follows that the following representation is valid

$$W^{p,\infty}(\Omega) = \bigcap_{n \in \mathbb{N}} \left(\widetilde{W^{p,\infty}(\Omega)}, \|\cdot\|_{p,n}\right),$$

where $(\widetilde{W^{p,\infty}}, \|\cdot\|_{p,n})$ is the completion of the normed space $(W^{p,\infty}(\Omega), \|\cdot\|_{p,n})$. It is well known that $(\widetilde{W^{p,\infty}(R^l)}, \|\cdot\|_{p,n}) = W_p^n(R^l)$. By virtue of what has been said, the space $W^{p,\infty}(\Omega)$ is a projective limit of sequences of Banach spaces $\{(\widetilde{W^{p,n}(\Omega)}, \|\cdot\|_{p,n})\}$. From the reflexivity of Sobolev spaces $W_p^n(\Omega)$ $(1 <$

$p < \infty$) it follows that the space $W^{p,\infty}(\Omega)$ $(1 < p < \infty)$ is totally reflexive Fréchet space. Let us now consider the Sobolev space of infinite order $W^{\infty}(a_\alpha, p)$, introduced by Yu. A. Dubinsky [46].

$W^{\infty}(a_\alpha, p)$ $(1 \le p < \infty,\ a_\alpha \ge 0)$ is Banach space and is defined as follows

$$W^{\infty}(a_\alpha, p) = \left\{ f \in W^{p,\infty}(\Omega);\ \sum_{|\alpha|=0}^{\infty} a_\alpha \|f^{(\alpha)}\|_p^p < \infty \right\}.$$

Here there are obtained imbedding theorems for infinite order Sobolev spaces of one into the other, then their imbeddings in Sobolev space of finite order. According to ([181], p. 31), the embedding of Banach space $(F, \|\cdot\|_1)$ into Banach space $(E, \|\cdot\|_2)$ is in many cases equivalent to set-theoretic inclusion $E \subset F$. A generalization of this result is valid for Fréchet spaces.

Proposition 2.6.7. *Let the Fréchet spaces $(E, \mathfrak{T}_1)$ and $(F, \mathfrak{T}_2)$ are subsets of the linear topological space L, and the convergence in the spaces $(E, \mathfrak{T}_1)$ and $(F, \mathfrak{T}_2)$ entails convergence in topology L. Let also some set $\Phi \subset E \cap F$ be dense in E and F. Then the set-theoretic inclusion $E \subset F$ is equivalent to the continuity of the inbedding of $(F, \mathfrak{T}_1)$ in $(E, \mathfrak{T}_2)$.*

Note that, generally speaking, as L, it is taken the space generalized functions $D'(\Omega)$. Since we are considering regular generalized functions, then as L we can take the space $L^1_{loc}(\Omega)$. Let $C^{\infty}(\overline{\Omega})$ denote the space of all infinitely differentiable functions f, derivatives of any order of which (including the function itself) admit continuous extensions to $\overline{\Omega}$, with a sequence of norms

$$\|f\|_n = \sup\{|f^{(\alpha)}(t)|;\ t \in \Omega,\ |\alpha| \le n\}, \quad n \in \mathbb{N}.$$

To prove the following Proposition 2.6.8 and Theorem 2.6.9, Proposition 2.6.7 and the well-known theorems of S. L. Sobolev and V. I. Kondrashov are applied ([55], p. 47).

Proposition 2.6.8. *The following statements are true:*

a) *For an arbitrary open set $\Omega \subset R^l$ the space $W^{p,\infty}(\Omega)$ is embedded in the Fréchet space $\mathcal{E}(\Omega)$.*

b) *If $\Omega \subset R^l$ is a bounded domain with a regular boundary in the sense of Calderon ([55], p. 45), then the space $W^{p,\infty}(\Omega)$ $(1 \le p < \infty)$ is isomorphic to the space $C^{\infty}(\overline{\Omega})$ and is the Fréchet-Schwartz space.*

Corollary. *If Ω is a bounded domain of class C^{∞} ([160], p. 300) with regular border or bounded domain with regular boundary satisfying only the Lifshitz condition or a domain of the type rectangular parallelepiped, then the space $W^{p,\infty}(\Omega)$*

is a nuclear space, i.e. for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the embedding $(\widetilde{W^{p,\infty}, \|\cdot\|_{p,m}}) \to (\widetilde{W^{p,\infty}, \|\cdot\|_{p,n}})$ *is a nuclear operator.*

Indeed, by virtue of Remark 1 ([160], p. 607) we obtain that under the conditions of the corollary of Proposition 2.6.8, the space $W^{p,\infty}(\Omega)$ is isomorphic to the nuclear Fréchet space s of rapidly decreasing sequences. Consequently, in this case in the Kondrashov's theorem for certain indices one can assert nuclearity of imbedding operators.

Theorem 2.6.9. *Let $\Omega \subset R^l$ be a bounded domain, $(1 \le p < \infty)$ and $a_\alpha > 0$ are an arbitrary sequence. Then the following statements are valid:*

a) *The spaces $W^\infty(a_\alpha, p)$ are embedded in the space $W^{p,\infty}(\Omega)$ and into the space $W_p^n(\Omega)$. Moreover, if $1 < p < \infty$, then these imbeddings are weakly completely continuous, i.e. images of some neighborhood at these mappings are weakly relatively compact.*

b) *If Ω is a bounded domain with a regular boundary and $1 < r < \infty$, $k \in \mathbb{N}$, then the following imbeddings*

$$W^\infty(a_\alpha, p) \to W^{p,\infty}(\Omega), \quad W^{p,\infty}(\Omega) \to W_r^k(\Omega) \ \ \textit{and} \ \ W^\infty(a_\alpha, p) \to W_r^k(\Omega)$$

are completely continuous, i.e. images of a certain neighborhood for these mappings are relatively compact.

Corollary. *If Ω satisfies the conditions of the Proposition* 2.6.8*, then for $1 \le p < \infty$, $1 < r < \infty$ and $k \in \mathbb{N}$ the imbeddings*

$$W^\infty(a_\alpha, p) \to W_r^k(\Omega) \quad \textit{and} \quad W^\infty(a_\alpha, p) \to W_r^k(\Omega)$$

are nuclear.

It should be noted that unit balls S_{a_α} $(a_\alpha > 0)$ of all spaces $W^\infty(a_\alpha, p)$, form a fundamental family of bounded absolutely convex sets in the space $W^{p,\infty}(\Omega)$. Indeed, every bounded absolutely convex subset B of space $W^{p,\infty}(\Omega)$ is contained in the ball S_{a_α} of space $W^\infty(a_\alpha, p)$, where $a_\alpha = M_\alpha^{-1} 2^{-n-1} e^{-n}$ for $|\alpha| = n$ and $M_\alpha = \sup\{\|f^{(\alpha)}\|_p^p;\ f \in B\}$.

Let us denote by $\overset{\circ}{W}{}^{p,\infty}(\Omega)$ $(1 \le p < \infty)$ the closure of a set of infinitely differentiable functions with compact supports $C_0^\infty(\Omega)$ in space $W^{p,\infty}(\Omega)$. The space $\overset{\circ}{W}{}^{p,\infty}(\Omega)$ is complete countable-normed space and for it is valid the representation

$$\overset{\circ}{W}{}^{p,\infty}(\Omega) = \bigcap_{n=0}^{\infty} \overset{\circ}{W}{}_p^n(\Omega)\,,$$

where $\overset{\circ}{W}{}^n_p(\Omega)$, as usual, is the completion of the space $C_0^\infty(\Omega)$ according to the norm $\|\cdot\|_{p,n}$. From the above, in particular, it follows that $\overset{\circ}{W}{}^n_2(R^l) = W^n_2(R^l)$ and therefore $\overset{\circ}{W}{}^{2,\infty}(R^l) = W^{2,\infty}(R^l)$.

Proposition 2.6.10. *The following statements are true:*

a) *If* Ω *is a bounded domain, then the space* $\overset{\circ}{W}{}^{p,\infty}(\Omega)$ $(1 \leq p < \infty)$ *is a nuclear space.*

b) *If the domain* Ω *satisfies the conditions of the corollary of Proposition 2.6.8, then the space* $\overset{\circ}{W}{}^{p,\infty}(\Omega)$ $(1 \leq p < \infty)$ *is nuclear, and therefore for every* $n \in \mathbb{N}$ *there exist* $m \in \mathbb{N}$ *such that the embedding*

$$\overset{\circ}{W}{}^m_p(\Omega) \to \overset{\circ}{W}{}^n_p(\Omega)$$

is nuclear.

c) *There is an unbounded domain* $\Omega \subset R^l$ *such that the space* $\overset{\circ}{W}{}^{2,\infty}(\Omega)$ *is nuclear.*

Indeed, the statement a) follows from the famous Rellich lemma and from the nuclearity embedding $\overset{\circ}{W}{}^m_p(\Omega) \to \overset{\circ}{W}{}^n_p(\Omega)$ when $m - n > l$. Statement b) follows from the nuclearity of the subspace $\overset{\circ}{W}{}^{p,\infty}(\Omega)$ of the space $W^{p,\infty}(\Omega)$. In [200], the conditions are indicated under which the operators $\overset{\circ}{W}{}^m_p(\Omega) \to \overset{\circ}{W}{}^n_p(\Omega)$ in case of unbounded domain Ω are the Hilbert-Schmidt operators. Therefore, for such a region Ω the space $\overset{\circ}{W}{}^{2,\infty}(\Omega)$ is also nuclear space due to the nuclearity of the product of Hilbert-Schmidt maps.

2.7 On homomorphisms, open and their adjoint operators

The operator between locally convex spaces E and F is called open if it maps open subsets of E into open subsets of F. A continuous linear operator between locally convex spaces which is open into its image, is called a homomorphism.

In Section 2.3, when studying the dual characterization of quojection, it has been proved that the adjoint operator to the canonical homomorphism of quojections on a Banach space is a strong homomorphism. In connection with this, we recall that in [65] two examples of homomorphisms of Fréchet spaces were built, whose adjoints are not strong homomorphisms, i.e. adjoint mappings are not homomorphisms, when dual spaces are endowed with the strong topologies. In particular, one of these examples is the canonical homomorphism of the Fréchet-Montel space onto the Banach space ℓ^1. These examples are also studied in detail in 2.7.4, giving the original and dual spaces with the different topologies.

Similar tasks often arise in applications and were intensively studied, starting with S. Banach, for certain spaces and topologies. The most important results were obtained in the works of J. Dieudonne [42], J. Dieudonne and L. Schwartz [43], A. Grothendieck [65], G. Köthe [80–83], F. Browder [29], V. S. Retakh [141], V. P. Palamodov [119], K. Floret and V. B. Moscatelli [52], S. Dierolf and D. Zarnadze [40], D. Zarnadze [204], J. Bonet and J. A. Conejero [22]. So far known results in this direction are most fully presented in monographs [50, 83, 134], but they have fragmented character. Special mention should be made of the strong homomorphisms and the strong adjoints to homomorphisms, i.e. about those cases when the original and dual spaces are endowed with the strong topologies. Question about them was studied in ([43], paragraph 12).

Grothendieck's homomorphism theorem ([83], p. 8). *Let $(E, \mathfrak{T}_1)$ and $(F, \mathfrak{T}_2)$ be locally convex spaces, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the families of equicontinuous sets in E' and F', respectively. The weak homomorphism A of the space $(E, \sigma(E, E'))$ into the space $(F, \sigma(F, F'))$ is a homomorphism of the space $(E, \mathfrak{T}_1)$ in the space $(F, \mathfrak{T}_2)$ if and only if $A'(\mathfrak{M}_2) = \mathfrak{M}_1 \cap A'(F')$, where A' is the adjoint operator and $\mathfrak{M}_1 \cap A'(F') = \{M \in \mathfrak{M}_1; M \subset A'(F')\}$.*

It should be noted, however, that this theorem is not valid for the topologies which are not compatible with the dualities $\langle E, E'\rangle$ and $\langle F, F'\rangle$. In particular, the above Theorem is not valid for the topology of uniform convergence $\mathfrak{T}_1 = \mathfrak{T}_{\mathfrak{M}_1}$ and $\mathfrak{T}_2 = \mathfrak{T}_{\mathfrak{M}_2}$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ families of weakly bounded sets E' and F' accordingly. Namely, it is not valid for the strong topologies, i.e. when $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are families of all weakly bounded sets in E' and F' respectively. Counterexample given in ([83], p. 10) for the last case shows that there is a weak homomorphism K of the LCS E onto the same space F, for which $K'(\mathfrak{M}_2) = K'(F') \cap \mathfrak{M}_1$, however K is not a strong homomorphism. The complexity of studying strong homomorphisms is also noted in ([83], p. 10). This is due to the fact that until now actually did not exist necessary and sufficient conditions in order for the weak homomorphism to be also a strong homomorphism and, in general, homomorphism in topologies, that are not compatible to the dualities $\langle E, E'\rangle$ and $\langle F, F'\rangle$.

This chapter proves a generalization of Grothendieck's theorem for topologies, which, in particular, are not compatible with the dualities. From here we derive necessary and sufficient conditions for a weak homomorphism A (respectively its adjoint mapping A', respectively its the second adjoint mapping A'') was again a homomorphism under endowments of the original (respectively dual, respectively bidual) spaces with strong topologies, Mackey topologies and topologies of the strong precompact convergence. Our approach is uniform and is that the homomorphicity of A (resp. A', resp. A'') in various situations characterized by the coincidence of two naturally generated topologies on the quotient space $E/\operatorname{Ker} A$ (resp. $F'/\operatorname{Ker} A'$, resp. $E''/\operatorname{Ker} A''$) and on the image $A(E)$ (resp. $A'(F')$, resp. $A''(F'')$). Similar reasoning yields a necessary and sufficient condition for a weak homomorphism to be also a homomorphism under the endowment of spaces with associated bornological topologies.

Here are the classes pairs of LCS E and F for which an arbitrary weak homomorphism is again a homomorphism in the above topologies. In particular, in sufficient detail are studied strong homomorphisms and strong adjoints to homomorphisms and strengthenings and generalizations of a few well-known results. By applying the results of F.Browder from [29] the conditions of openness and strong openness of weakly open operators having closed graphs are also given.

The theorems on homomorphisms given in this section are the result of a unified approach to their study. In particular, special diagrams are discussed for the first time in the 100-year history of homomorphism research. On them the dependence between the topologies on the quotient space and on the image of the operator, which is obtained as a result of the canonical decomposition of operator, are discussed. They give possibility of comparison these topologies and draw the necessary conclusions.

2.7.1 Homomorphisms between locally convex spaces

This section provides conditions for a homomorphism between the LCS E and F would again be a homomorphism in different known topologies of the spaces E and F.

Let E be an LCS and let $\mathfrak{M}$ be some family weakly bounded sets of the dual space E', containing the class of all finite sets δ. $\mathfrak{T}_{\mathfrak{M}}(E')$ denotes the topology on E that is uniform convergence on sets from $\mathfrak{M}$. In particular, through $\beta(E,E')$ (resp. $\sigma(E,E')$, resp. $\tau(E,E')$, resp. $\mathfrak{T}_c(E')$) will denote the topology on E of uniform convergence on all bounded (resp. on finite, resp. on absolutely convex and $\sigma(E,E')$-compact sets, resp. on strongly precompact) sets from E'.

By symmetry, there are defined the topologies of the uniform convergence $\beta(E',E)$, $\sigma(E',E)$, $\tau(E',E)$ and $\mathfrak{T}_c(E)$ on E'. For a subspace G of the locally convex space $(E,\mathfrak{T}_{\mathfrak{M}}(E'))$, by $\mathfrak{T}_{\mathfrak{M}}(E')\cap G$ it is denoted the topology induced on G, and through $\mathfrak{T}_{\widehat{\mathfrak{M}}}(E'/G^{\perp})$ the topology of uniform convergence on sets $\widehat{\mathfrak{M}}=k(\mathfrak{M})=\{k(M);\ M\in\mathfrak{M}\}$, where k is the canonical mapping of E' to $E'/G^{\perp}$. Next, for a closed subspace G of the space $(E,\mathfrak{T}_{\mathfrak{M}}(E'))$ via $\mathfrak{T}_{\mathfrak{M}}(E')/G$ denotes the quotient topology on E/G, and by $\mathfrak{T}_{\bar{\mathfrak{M}}}(G^{\perp})$ the topology on E/G of the uniform convergence on the family $\bar{\mathfrak{M}}=\{M\in\mathfrak{M};\ M\subset G^{\perp}\}$.

Theorem 2.7.1. *Let E and F be the locally convex spaces with the saturated classes of weakly bounded subsets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the dual spaces E' and F', respectively. The weak homomorphism A of the space E in the space F is the homomorphism of the space $(E, \mathfrak{T}_{\mathfrak{M}_1}(E'))$ in the space $(F, \mathfrak{T}_{\mathfrak{M}_2}(F'))$ if and only if the following conditions are satisfied:*

a)$\overline{\mathfrak{M}}_1 = \widetilde{\widehat{\mathfrak{M}}}_2$ *and* $A'(\mathfrak{M}_2) \subset \mathfrak{M}_1$, *where* A' *is dual operator to* A, $\overline{\mathfrak{M}}_1 = \{M \in \mathfrak{M}_1;\ M \subset \operatorname{Ker} A^{\perp}\}$, $\widehat{\mathfrak{M}}_2 = K_1(\mathfrak{M}_2) = \{K_1(M);\ M \in \mathfrak{M}_2\}$, $K_1 : F' \to F'/A(E)^{\perp}$ *is a canonical map, and* $\widetilde{\widehat{\mathfrak{M}}}_2$ *is saturated hull of* $\widehat{\mathfrak{M}}_2$.

b) $\mathfrak{T}_{\mathfrak{M}_1}(E')/\operatorname{Ker} A = \mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp})$ *on* $E/\ Ker A$, *where* $\mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp})$ *is a topology of uniform convergence on the class* $\bar{\mathfrak{M}}_1$.

c) $\mathfrak{T}_{\mathfrak{M}_2}(F') \cap A(E) = \mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})$ *on* $A(E)$, *where* $\mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})$ *is the uniform convergence topology on* $\widehat{\mathfrak{M}}_2$.

Proof. *Sufficiency*. Let $A = J\breve{A}K$ be the canonical decomposition of the weak homomorphism A, where K is the canonical homomorphism of the space $(E, \sigma(E, E'))$ on $(E/\operatorname{Ker} A, \sigma(E, E')/\operatorname{Ker} A)$, $\breve{A}$ is weak isomorphism of $(E/\operatorname{Ker} A, \sigma(E, E')/\operatorname{Ker} A)$ onto $(A(E), \sigma(F, F') \cap A(E))$ and J is monomorphism $(A(E), \sigma(F, F') \cap A(E))$ into $(F, \sigma(F, F'))$. Due to the known properties of the weak topology ([82], p. 276), the equalities $\sigma(E, E')/\operatorname{Ker} A = \sigma(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})$ on $E/\operatorname{Ker} A$ and $\sigma(F, F') \cap A(E) = \sigma(A(E),\ F'/A(E)^{\perp})$ are valid on $A(E)$, since $(E/\operatorname{Ker} A)' = \operatorname{Ker} A^{\perp}$ and $A(E)' = F'/A(E)^{\perp}$. Therefore, the spaces $(E/\operatorname{Ker} A,\ \sigma(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp}))$ and $(A(E),\ \sigma(A(E),\ F'/A(E)^{\perp}))$ are isomorphic.

Hence from condition a) it follows that the spaces $(E/\operatorname{Ker} A, \mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp}))$ and $(A(E), \mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp}))$ are isomorphic, where $\mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp})$ is the topology on $E/\operatorname{Ker} A$, of uniform convergence on $\overline{\mathfrak{M}}_1$, and $\mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})$ is the topology on $A(E)$ uniform convergence on sets $K_1(\mathfrak{M}_2) \subset F'/A(E)^{\perp}$. From the conditions $A'(\mathfrak{M}_2) \subset \mathfrak{M}_1$ by virtue of Theorem 1 ([83], p. 3), it follows continuity of the operator A from the space $(E,\ \mathfrak{T}_{\mathfrak{M}_1}(E'))$ into $(F, \mathfrak{T}_{\mathfrak{M}_2}(F'))$. Next, from the conditions b) and c) it follows that $\breve{A}$ is also an isomorphism of the spaces $(E/\operatorname{Ker} A, \mathfrak{T}_{\mathfrak{M}_1}(E')/\operatorname{Ker} A)$ on $(A(E),\ \mathfrak{T}_{\mathfrak{M}_2}(F') \cap A(E))$, i.e. A is the homomorphism of the space $(E,\ \mathfrak{T}_{\mathfrak{M}_1}(E'))$ into $(F,\ \mathfrak{T}_{\mathfrak{M}_2}(F'))$.

Necessity. The weak homomorphism A of the space E in F is also the homomorphism of the space $(E, \mathfrak{T}_{\mathfrak{M}_1}(E'))$ into $(F,\ \mathfrak{T}T_{\mathfrak{M}_2}(F'))$. As stated above, $\breve{A}$ is the weak isomorphism of the space $(E/\operatorname{Ker} A,\ \sigma(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp}))$ onto $(A(E), \sigma(A(E), F'/A(E)^{\perp}))$. In these spaces also coincide $\mathfrak{T}_{\mathfrak{M}_1}(E')/\operatorname{Ker} A$ and $\mathfrak{T}_{\mathfrak{M}_2}(F') \cap A(E)$, since A is the homomorphism of the space $(E, \mathfrak{T}_{\mathfrak{M}_1}(E'))$ in $(F, \mathfrak{T}_{\mathfrak{M}_2}(F'))$. Hence, again by virtue of Theorem 1 from ([83], p. 3) we obtain $A'(\mathfrak{M}_2) \subset \mathfrak{M}_1$. This is equivalent to the fact that the mapping $\breve{A}$ is continuous, since it is the mapping of the space $(E/\operatorname{Ker} A,\ \mathfrak{T}_{\widehat{\mathfrak{M}_1}}(\operatorname{Ker} A^{\perp}))$ onto $(A(E),\ ;\mathfrak{T}_{\widehat{\mathfrak{M}_2}}(F'/A(E)^{\perp}))$.

Indeed, $A' = K'\check{A}'J'$, where $K',\ \check{A}'$ and J' are the adjoints to the mappings $K,\ \check{A}$ and J, respectively. In that case we have $K'\check{A}'J'(\mathfrak{M}_2) \subset \mathfrak{M}_1$, but as is known, $J'(\mathfrak{M}_2) = K_1(\mathfrak{M}_2) = \widehat{\mathfrak{M}_2}$, and therefore $K'\check{A}'(\widehat{\mathfrak{M}_2}) \subset \mathfrak{M}_1$, i.e. $\check{A}'(\widehat{\mathfrak{M}_2}) \subset K'^{(-1)}(\mathfrak{M}_1) = \mathfrak{M}_1 \cap \operatorname{Ker} A^{\perp} = \overline{\mathfrak{M}}_1$. Further, by virtue of Theorem 1 from ([82], p. 276), we obtain the inequality $\mathfrak{T}_{\mathfrak{M}_2}(F') \cap A(E) \leq \mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})$ on $A(E)$. Indeed, for $M \in \mathfrak{M}_2$ the following equalities are valid: $K_1(M)^{0A(E)} = J'(M)^{0A(E)} = J^{-1}(M^{0F}) = M^{0F} \cap A(E)$. The above inequality is true, since in the saturated cover of $\widehat{\mathfrak{M}}_2$, there may occur the sets that are not contained in the K_1-image of the sets from $\mathfrak{M}_2$. The inequality $\mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp}) \leq \mathfrak{T}_{\mathfrak{M}_1}(E')/\operatorname{Ker} A$ can be proved in a similar manner using Theorem 3 ([82], p. 277). On account of the preceding arguments, the following diagram is valid:

$$
\begin{tikzcd}
(E/\operatorname{Ker} A,\ \mathfrak{T}_{\mathfrak{M}_1}(E')/\operatorname{Ker} A) \arrow[r, shift left, "\check{A}"] \arrow[d] & (A(E),\ \mathfrak{T}_{\mathfrak{M}_2}(F') \cap A(E)) \arrow[l, shift left] \\
(E/\operatorname{Ker} A,\ \mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp})) \arrow[r, "\check{A}"] & (A(E),\ \mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})) \arrow[u]
\end{tikzcd}
$$

where vertical arrows denote the continuous identical algebraic isomorphisms. From this it turns out that the indicated two topologies on $E/\operatorname{Ker} A$ and $A(E)$ coincide. Furthermore, the topologies $\mathfrak{T}_{\bar{\mathfrak{M}}_1}(\operatorname{Ker} A^{\perp})$ and $\mathfrak{T}_{\widehat{\mathfrak{M}}_2}(F'/A(E)^{\perp})$ on the weakly isomorphic spaces $E/\operatorname{Ker} A$ and $A(E)$ also coincide. By Theorem 4 ([82], p. 256) the saturated covers of the classes $A'(\mathfrak{M}_2)$ and $\overline{\mathfrak{M}}_1$ also coincide, i.e. condition a) is fulfilled too.

Note that if the classes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ consist only of equicontinuous sets, then the class $\widehat{\mathfrak{M}}_2$ is also saturated and the conditions b) and c) are always fulfilled by virtue of Theorems 1 and 2 ([82], pp. 275–276). As the above-mentioned example from [83] shows, we do not have the same situation in the general case. Below we will give the examples of the topologies for which the statements b) and c) imply a). $\square$

Corollary. *Let in the notation of Theorem* 2.7.1 A *be a weak monomorphism of* E *in* F. *Then* A *is a monomorphism of* $(E, \mathfrak{T}_{\mathfrak{M}_1})$ *into* $(F, \mathfrak{T}_{\mathfrak{M}_2})$ *if and only if* $A'(\mathfrak{M}_2) \subset \mathfrak{M}_1$ *and* $\overline{\mathfrak{M}}_1 \subset \widetilde{A'(\mathfrak{M}_2)}$, *where* $\widetilde{A'(\mathfrak{M}_2)}$ *consists of weak closure sets of the form* $A'(M),\ M \in \mathfrak{M}_2$ *in* $F'/A(E)^{\perp}$ *with respect to the dual pair* $\langle A(E), F'/A(E)^{\perp}\rangle$.

Indeed, if A is a monomorphism of $(E,\ \mathfrak{T}_{\mathfrak{M}_1})$ in $(F,\ \mathfrak{T}_{\mathfrak{M}_2})$, then the conditions of the corollary follow from Theorem 2.7.1. Vice versa, from the condition $A'(\mathfrak{M}_2)$

$\subset \mathfrak{M}_1$, by virtue of Theorem 1 from ([83], p. 3), we obtain the continuity of A as a mapping from $(E,\ \mathfrak{T}_{\mathfrak{M}_1})$ to $(F,\ \mathfrak{T}_{\mathfrak{M}_2})$. By condition, A is a weak monomorphism, so $A'(F') = E' = \operatorname{Ker} A^{\perp},\ \overline{\mathfrak{M}}_1 = \mathfrak{M}_1$, and repeating the reasoning given in the proof of Theorem 1 ([83], p. 10), we obtain $\mathfrak{T}_{\mathfrak{M}_1} - \mathfrak{T}_{\mathfrak{M}_2}$ openness of operator A.

Let us now give a specification of Theorem 2.7.1 for known topologies.

Theorem 2.7.2. *Let A be a weak homomorphism of the LCS E in the LCS F. A is a strong homomorphism, i.e. a homomorphism of the space $(E,\ \beta(E,E'))$ in the space $(F,\ \beta(F,F'))$, if and only if $\beta(E,E')/\operatorname{Ker} A = \beta(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})$ on $E/\operatorname{Ker} A$ and $\beta(F,F') \cap A(E) = \beta(A(E),\ F'/A(E)^{\perp})$ on $A(E)$, where $\beta(F,F')/\operatorname{Ker} A$ is quotient topology of the strong topology $\beta(E,E')$, $\beta(F,F') \cap A(E)$ is induced topology on $A(E)$, $\beta(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})$ and $\beta((A(E),\ F'/A(E)^{\perp})$ are strong topologies of dual pairs $\langle E/\operatorname{Ker} A, \operatorname{Ker} A^{\perp}\rangle$ and $\langle A(E), F'/A(E)^{\perp}\rangle$, respectively.*

Proof. *Sufficiency*. By repeating the reasoning, which were given in the proof of Theorem 2.7.1, taking into account the fact that a weakly continuous operator is strongly continuous, we obtain that the following diagram holds:

$$\begin{tikzcd}
 & (E/\operatorname{Ker} A,\ \beta(E,E')/\operatorname{Ker} A) \arrow[r, "\check{A}"] \arrow[dd] & (A(E),\ \beta(F,F') \cap A(E)) \arrow[dr] & \\
(E,\ \beta(E,E')) \arrow[ur] \arrow[dr] & & & (F,\ \beta(F,F')) \\
 & (E/\operatorname{Ker} A,\ \beta(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})) \arrow[r, shift left, "\check{A}"] & (A(E),\ \beta(A(E),\ F'/A(E)^{\perp})) \arrow[l, shift left] \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

It follows from the condition that the continuous algebraic isomorphisms indicated by vertical arrows are topological isomorphisms. Therefore, A is a strong homomorphism.

Necessity. We again use the above diagram. In this case, the operator indicated by the upper horizontal arrow becomes a topological isomorphism. From this, we immediately get the coincidence of the indicated topologies on $E/\operatorname{Ker} A$ and $A(E)$. □

It should be noted that in the proof of the necessity of Theorem 2.7.2, the a priori conditions imposed on A can be weakened. Namely, for the coincidence of the indicated topologies on $E/\operatorname{Ker} A$ and $A(E)$, it is enough to require that A is a weakly continuous strong homomorphism. However, one cannot expect that A will be a weak homomorphism.

Corollary. *If A is a homomoropism of the LCS of the barrelled space $(E, \mathfrak{T}_1)$ to LCS $(F, \mathfrak{T}_2)$, then A is a strong homomorphism.*

Indeed, the barrelledness of the space $(E, \mathfrak{T}_1)$ means that $\mathfrak{T}_1 = \beta(E, E')$. Since the quotient space of the barrelled space is barrelled, we have

$$\mathfrak{T}_1 / \operatorname{Ker} A = \beta(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp}) = \beta(E, E')/\operatorname{Ker} A$$

on $E/\operatorname{Ker} A$. By condition, the quotient space $(E/\operatorname{Ker} A,\ \beta(E, E')/\operatorname{Ker} A)$ is isomorphic to the subspace $(A(E),\ \mathfrak{T}_2 \cap A(E))$ of the space $(F, \mathfrak{T}_2)$. Therefore, the last space is also a barrelled space and the following equality holds:

$$\mathfrak{T}_2 \cap A(E) = \beta(F, F') \cap A(E) = \beta(A(E),\ F'/A(E)^{\perp}).$$

By $\mathfrak{T}_{b^*}(E')$ it is denoted the topology on E-uniform convergence on strongly bounded sets of E', i.e. topology of uniform convergence on bounded sets of the space $(E', \beta(E, E'))$.

Theorem 2.7.3. *Let A be a weak homomorphism of the LCS E into the LCS F. A is a $\mathfrak{T}_{b^*}$-homomorphism, i.e. homomorphism of LCS $(E, \mathfrak{T}_{b^*}(E'))$ in the LCS $(F, \mathfrak{T}_{b^*}(F'))$ if and only if $\mathfrak{T}_{b^*}(E')/\operatorname{Ker} A = \mathfrak{T}_{b^*}(\operatorname{Ker} A^{\perp})$ on $E/\operatorname{Ker} A$ and $\mathfrak{T}_{b^*}(F') \cap A(E) = \mathfrak{T}_{b^*}(F'/A(E)^{\perp})$ on $A(E)$.*

Corollary. *If A is a homomorphism of the quasi-barrelled LCS E in the LCS F, then A is a $\mathfrak{T}_{b^*}$-homomorphism.*

The proofs of these statements are almost verbatim repetition of the above reasoning, due to the known properties of $\mathfrak{T}_{b^*}$-topologies, and we omit it.

Let us now present the conditions for a weak homomorphism to be a homomorphism in Mackey topologies.

Theorem 2.7.4. *Let A be a weak homomorphism of the LCS E into the LCS F. A is a $\mathfrak{T}_k$-homomorphism, i.e. a homomorphism of the space $(E, \tau(E, E'))$ in the space $(F, \tau(F, F'))$ if and only if $\tau(F, F') \cap A(E) = \tau(A(E), F'/A(E)^{\perp})$ on $A(E)$, where $\tau(F, F') \cap A(E)$ is the induced topology on $A(E)$ and $\tau(A(E), F'/A(E)^{\perp})$ is the Mackey topology on $A(E)$ with respect to the dual pair $\langle A(E), F'/A(E)^{\perp}\rangle$.*

The validity of this theorem follows from Theorem 2.7.1 with regard for the equality $\tau(E, E')/\operatorname{Ker} A = \tau(E/\operatorname{Ker} A, \operatorname{Ker} A^{\perp})$ on $E/\operatorname{Ker} A$, which is obtained by Theorem 3 ([82], p. 277). It should also be noted that when proving the necessity of the coincidence of the mentioned topologies on $A(E)$, it is sufficient only to require that A is $\mathfrak{T}_k$-homomorphism. Really, due to the weak continuity of $\mathfrak{T}_k$-continuous mappings ([83], p. 4.), it turns out that the following diagram is valid:

$$\begin{tikzcd}
(E/\operatorname{Ker} A,\ \tau(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})) \arrow[r] \arrow[d, shift right] & (A(E),\ \tau(A(E),\ F'/A(E)^{\perp})) \arrow[d] \\
(E/\operatorname{Ker} A,\ \tau(E,E')/\operatorname{Ker} A) \arrow[u, shift right] \arrow[r, shift left] & (A(E),\ \tau(F,F') \cap A(E)) \arrow[l, shift left]
\end{tikzcd}$$

where the arrows indicate continuous algebraic isomorphisms.

Corollary 1. *Let A be a continuous weak homomorphism LCS $(E, \mathfrak{T}_1)$ in LCS $(F, \mathfrak{T}_2)$ such that the space $(A(E), \mathfrak{T}_2 \cap A(E))$ is the Mackey space. Then A is a homomorphism of the space $(E, \mathfrak{T}_1)$ in the space $(F, \mathfrak{T}_2)$.*

Indeed, due to the above results, from the condition we find that the following diagram holds:

$$\begin{tikzcd}
(E/\operatorname{Ker} A,\ \mathfrak{T}_1/\operatorname{Ker} A) \arrow[r] & (A(E),\ \mathfrak{T}_2 \cap A(E)) \arrow[r, shift left] & (A(E),\ \tau(A(E), F'/A(E)^{\perp})) \arrow[l, shift left] \arrow[d, shift right] \\
(E/\operatorname{Ker} A,\ \tau(E,E')/\operatorname{Ker} A) \arrow[u] \arrow[rr, shift left] & & (E/\operatorname{Ker} A,\ \tau(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp}) \arrow[ll, shift left] \arrow[u, shift right]
\end{tikzcd}$$

where the arrows indicate continuous algebraic isomorphisms. This implies that A is a homomorphism of the space $(E, \mathfrak{T}_1)$ in the space $(F, \mathfrak{T}_2)$.

This result generalizes the first part of Proposition 5 of ([83], p. 8).

Corollary 2. *Let A be a weak homomorphism of the barrelled (resp. quasi-barrelled) LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$ such that the space $(A(E), \mathfrak{T}_2 \cap A(E))$ is the Mackey space. Then A is a strong homomorphism (resp. $\mathfrak{T}_{b^*}$-homomorphism).*

Proof. Let us first prove that A is continuous. Really, from the weak continuity of A it follows that it is strongly continuous (resp. $\mathfrak{T}_{b^*}$-continuous). Further, it is continuous as a mapping from $(E, \beta(E, E'))$ to $(F, \mathfrak{T}_2)$ (resp. as a mapping from $(E, \mathfrak{T}_{b^*}(E'))$ to $(F, \mathfrak{T}_2)$, since $\mathfrak{T}_1 = \beta(E, E')$ and $\mathfrak{T}_2 \leq \beta(F, F')$ (resp. $\mathfrak{T}_1 = \mathfrak{T}_{b^*}(E')$ and $\mathfrak{T}_2 \leq \mathfrak{T}_{b^*}(F')$). By Corollary 1, A is a homomorphism, and by virtue of the corollary of Theorem 2.7.2 (resp. by the corollary of Theorem 2.7.3), A is the strong homomorphism (resp. A is the $\mathfrak{T}_{b^*}$-homomorphism). This result generalizes Sentence 21 from [42]. □

By $\mathfrak{T}_c(E')$ it is denoted the topology on E of uniform convergence on strongly precompact sets of the space E'.

Theorem 2.7.5. *Let A be a weak homomorphism of the LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$. A is a $\mathfrak{T}_c$-homomorphism, i.e. a homomorphism of the space $(E,\ \mathfrak{T}_c(E'))$ in the space $(F,\ \mathfrak{T}_c(F'))$ if and only if $\mathfrak{T}_c(E')/\operatorname{Ker} A = \mathfrak{T}_c(\operatorname{Ker} A^{\perp})$ on $E/\operatorname{Ker} A$ and $\mathfrak{T}_c(F') \cap A(E) = \mathfrak{T}_c(F'/A(E)^{\perp})$ on $A(E)$.*

This theorem can be proved similarly to the above theorems taking into account the fact that the weakly continuous operator A is $\mathfrak{T}_c$-continuous, since its adjoint map A' is strong continuous and A'-image of a strongly precompact set in F' is the same in E'.

It should also be noted that, by virtue of Theorem 1 from ([82], p. 276), the topologies of $\mathfrak{T}_c(E') \cap A(E)$ and $\mathfrak{T}_c(F'/A(E)^{\perp})$ on $A(E)$ coincide if every precompact subset of the quotient space $(F'/A(E)^{\perp}, \beta(F', F)/A(E)^{\perp})$ is contained in the closure of the canonical image of the precompact sets of the space $(F',\ \beta(F', F))$.

There are the examples of monomorphisms of the spaces of test functions D and generalized functions D', on whose range the considered in Theorem 2.7.5 two topologies are different. It is well known that the spaces D and D' are related by the "Pontryagin duality", i.e. the topology of each of these spaces coincides with a uniform convergence topology on strongly compact sets of the dual space. As was shown in [151], this duality does not any longer extend to the quotient spaces of the spaces D and D'. This means that there exists a closed subspace G of D (resp. a closed subspace M of the space D') such that the spaces $(G, \mathfrak{T}_c(D') \cap G)$ and $(G,\ \mathfrak{T}_c(D'/G^{\perp}))$ (resp. the spaces $(M,\ \mathfrak{T}_c(D) \cap M)$ and $(M,\ \mathfrak{T}_c(D/M^{\perp}))$ are not isomorphic. By virtue of Theorem 2.7.5, this is equivalent to the fact that the monomorphism $J : G \to D$ (resp. the monomorphism $J_1 : M \to D'$) is not a $\mathfrak{T}_c$-monomorphism.

Consider now this problem assuming that the spaces E and F are equipped with the so-called associated bornological topologies. Let $(E, \mathfrak{T})$ be an LCS. By ([82], p. 380), there exists, on E, the strongest locally convex topology $\mathfrak{T}^{\times}$, which possesses the same bounded sets as the topology $\mathfrak{T}$. For the topology $\mathfrak{T}^{\times}$, the basis of neighborhoods of zero consists of all absolutely convex sets which absorb all bounded sets. $(E, \mathfrak{T}^{\times})$ is a bornological space and is called the associated bornological space.

Theorem 2.7.6. *Let A be a weak homomorphism of the LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$. Then A is a $\mathfrak{T}^{\times}$-homomorphism, i.e. a homomorphism of the space $(E, \mathfrak{T}_1^{\times})$ in $(F, \mathfrak{T}_2^{\times})$ if and only if $\mathfrak{T}_1^{\times}/\operatorname{Ker} A = (\mathfrak{T}_1/\operatorname{Ker} A)^{\times}$ on $E/\operatorname{Ker} A$ and $\mathfrak{T}_2^{\times} \cap A(E) = (\mathfrak{T}_2 \cap A(E))^{\times}$ on $A(E)$, where $\mathfrak{T}_1^{\times}/\operatorname{Ker} A$ is the quotient topology of the topology $\mathfrak{T}_1^{\times}$, $(\mathfrak{T}_1/\operatorname{Ker} A)^{\times}$ is the associated bornological topology of the quotient topology $\mathfrak{T}_1/\operatorname{Ker} A$, $\mathfrak{T}_2^{\times} \cap A(E)$ is the induced topology on $A(E)$ of the* *topology $\mathfrak{T}_2^{\times}$, and $(\mathfrak{T}_2 \cap A(E))^{\times}$ is the associated bornological topology of the induced on $A(E)$ topology $\mathfrak{T}_2 \cap A(E)$.*

Proof. First, we prove that each weakly continuous mapping A is a continuous mapping of the space $(E, \mathfrak{T}_1^{\times})$ in $(F, \mathfrak{T}_2^{\times})$. This follows from the local boundedness of the mapping A, i.e. A transforms the bounded sets from $(E, \mathfrak{T}_1)$ into the same kind of subsets in $(F, \mathfrak{T}_2)$. Therefore, the following diagram is valid:

$$\begin{tikzcd}
& (E/\operatorname{Ker} A,\ \sigma(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})^{\times}) \arrow[r, shift right] & (A(E),\ \sigma(A(E),\ F'/A(E)^{\perp})^{\times}) \arrow[l, shift right] \arrow[dr] & \\
(E,\ \sigma(E, E')^{\times}) \arrow[ur] \arrow[dr] & & & (F,\ \sigma(F, F')^{\times}) \\
& (E/\operatorname{Ker} A,\ \sigma(E, E')^{\times}/\operatorname{Ker} A) \arrow[r] & (A(E),\ \sigma(F, F')^{\times} \cap A(E)), \arrow[ur] &
\end{tikzcd}$$

where the arrows denote the continuous mappings. Since the associated bornological topology depends only on the dual pair, the topologies $\sigma(E, E')$, $\sigma(E/\operatorname{Ker} A,\ \operatorname{Ker} A^{\perp})$, $\sigma(A(E), F'/A(E)^{\perp})$ and $\sigma(F, F')$ in the above diagram can be replaced by the topologies $\mathfrak{T}_1,\ \mathfrak{T}_1/\operatorname{Ker} A,\ \mathfrak{T}_2 \cap A(E)$ and $\mathfrak{T}_2$, respectively.

Let us now prove the continuity of the identity mappings

$$(E/\operatorname{Ker} A,\ \mathfrak{T}_1^{\times}/\operatorname{Ker} A) \to (E/\operatorname{Ker} A,\ (\mathfrak{T}_1/\operatorname{Ker} A)^{\times}),$$

$$(A(E),\ (\mathfrak{T}_2 \cap A(E))^{\times}) \to (A(E),\ \mathfrak{T}_2^{\times} \cap A(E)).$$

Indeed, let W be a $(\mathfrak{T}_1/\operatorname{Ker} A)^{\times}$-neighborhood, then W absorbs all $\mathfrak{T}_1/\operatorname{Ker} A$-bounded sets of quotient space $E/\operatorname{Ker} A$. Therefore, the set $K^{-1}(W)$ absorbs all $\mathfrak{T}_1$-bounded sets in E and is a $\mathfrak{T}_1^{\times}$-neighborhood, where $K : E \to E/\operatorname{Ker} A$ is the canonical mapping. This means that $W = K(K^{-1}W)$ is a $\mathfrak{T}_1^{\times}/\operatorname{Ker} A$-neighborhood. Let now U be $\mathfrak{T}_2^{\times} \cap A(E)$-neighborhood in $A(E)$, then $U = V \cap A(E)$, where V is $\mathfrak{T}_2^{\times}$-neighborhood and therefore V absorbs all bounded sets in $(F, \mathfrak{T}_2)$. Hence it immediately follows that $U = V \cap A(E)$ absorbs all bounded sets $A(E)$, i.e. U is $(\mathfrak{T}_2 \cap A(E))^{\times}$-neighborhood in $A(E)$. By virtue of this reasoning the following diagram is valid:

$$\begin{tikzcd}
& (E/\operatorname{Ker} A,\ (\mathfrak{T}_1/\operatorname{Ker} A)^{\times}) \arrow[r, shift left] & (A(E),\ (\mathfrak{T}_2 \cap A(E))^{\times}) \arrow[l, shift left] \arrow[dr] \arrow[dd] & \\
(E,\ \mathfrak{T}_1^{\times}) \arrow[ur] \arrow[dr] & & & (F,\ \mathfrak{T}_2^{\times}) \\
& (E/\operatorname{Ker} A,\ \mathfrak{T}_1^{\times}/\operatorname{Ker} A) \arrow[r] \arrow[uu] & (A(E),\ \mathfrak{T}_2^{\times} \cap A(E)) \arrow[ur] &
\end{tikzcd}$$

where the arrows denote continuous mappings. From this diagrams it is already easy to obtain a proof of our statement. □

2.7.2 Adjoint operator to homomorphism between locally convex spaces

It is well known that A is a weak homomorphism of the LCS E in the LCS F with the closed range if and only if the adjoint mapping A' is a weak homomorphism with the weakly closed range $A'(F') = \operatorname{Ker} A^{\perp}$. Then if $A = J\breve{A}K$ is the decomposition of the weak homomorphism A and J', $\breve{A}'$ and K' are the adjoint to the mappings J, $\breve{A}$ and K, respectively, then $A' = K'\breve{A}'J'$ is the natural decomposition of the weak homomorphism A'. This means that J' is a canonical homomorphism of the space $(F', \sigma(F', F))$ to $(A(E)', \sigma(A(E)', A(E))) = (F'/\operatorname{Ker} A', \sigma(F', F)/\operatorname{Ker} A')$, K' is a weak monomorphism of the space $((E/\operatorname{Ker} A)', \sigma((E/\operatorname{Ker} A)', E/\operatorname{Ker} A)) = (\operatorname{Ker} A^{\perp}, \sigma(E'E) \cap \operatorname{Ker} A^{\perp}) = (A'(F'), \sigma(E', E) \cap A'(F'))$ in $(E', \sigma(E', E))$, and $\breve{A}'$ is a weak isomorphism of the space $(F'/\operatorname{Ker} A', \sigma(F', F)\ /\operatorname{Ker} A')$ to $(A'(F'), \sigma(E', E) \cap A'(F'))$.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be saturated classes of bounded subsets in E and F, respectively. Denote by $\mathfrak{T}_{\mathfrak{M}_1}(E)$ and $\mathfrak{T}_{\mathfrak{M}_2}(F)$ the topologies on E' and F' of uniform convergence on the subsets from $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively. Let A be a weakly continuous mapping of the space E onto the space F. Then the adjoint mapping A' is a continuous mapping of the space $(F', \mathfrak{T}_{\mathfrak{M}_2}(F))$ in $(E', \mathfrak{T}_{\mathfrak{M}_1}(E))$ if and only if $A(\mathfrak{M}_1) \subset \mathfrak{M}_2$. Hence it follows that the adjoint to the weak isomorphism of the space E in the space F is strong and $\mathfrak{T}_k$-isomorphism. The example of the identical mapping I of the normed space $(E, \|\cdot\|)$ on $(E, \sigma(E, E'))$ shows that its adjoint I' is a strong isomorphism, but I is not an isomorphism.

Theorem 2.7.7. *Let E and F be the locally convex spaces, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the saturated classes of bounded sets in E and F, respectively, and $A : E \to F$ be a weak homomorphism with closed range $A(E)$. The adjoint mapping A' is a homomorphism of the space $(F', \mathfrak{T}_{\mathfrak{M}_2}(F))$ in the space $(E', \mathfrak{T}_{\mathfrak{M}_1}(E))$ if and only if*

a) $A(\mathfrak{M}_1) \subset \mathfrak{M}_2$ *and* $\overline{\mathfrak{M}}_2 \subset \widetilde{\widehat{\mathfrak{M}}}_1$, *where* $\overline{\mathfrak{M}}_2 = \{M \in \mathfrak{M}_2;\ M \subset A(E)\} = \mathfrak{M}_2 \cap A(E)$, $\widehat{\mathfrak{M}}_1 = K(\mathfrak{M}_1) = \{K(M);\ M \in \mathfrak{M}_1\}$, $\widetilde{\widehat{\mathfrak{M}}}_1$ *is the saturated cover* $\widehat{\mathfrak{M}}_1$.

b) $\mathfrak{T}_{\widehat{\mathfrak{M}}_1}(E/\operatorname{Ker} A) = \mathfrak{T}_{\mathfrak{M}_1}(E) \cap A'(F')$ *on* $(E/\operatorname{Ker} A)' = \operatorname{Ker} A^{\perp} = A'(F')$, *where* $\mathfrak{T}_{\widehat{\mathfrak{M}}_1}$ *is the topology of uniform convergence on the sets from* $\widehat{\mathfrak{M}_1}$.

c) $\mathfrak{T}_{\mathfrak{M}_2}(F)/\operatorname{Ker} A' = \mathfrak{T}_{\overline{\mathfrak{M}}_2}(A(E))$ *on* $A(E)' = F'/A(E)^{\perp} = F'/\operatorname{Ker} A'$.

The validity of this theorem follows from Theorem 2.7.1 if the latter is applied to the weak homomorphism A' of the space $(F', \sigma(F', F))$ in the space

$(E', \ \sigma(E', E))$. One can also prove it in a straightforward manner analogously to the proof of Theorem 2.7.1.

Next, we give a few concrete reformulations of Theorem 2.7.7 for the most important topologies.

Theorem 2.7.8. *Let A be a weak homomorphism of the LCS E in the LCS F with a weak closed range. A' is a strong homomorphism, i.e. a homomorphism of the space $(F', \beta(F', F))$ in the space $(E', \beta(E', E))$ if and only if $\beta(F', F)/\operatorname{Ker} A' = \beta(F'/\operatorname{Ker} A', A(E))$ on $F'/\operatorname{Ker} A'$ and $\beta(\operatorname{Ker} A^{\perp}, E/\operatorname{Ker} A) = \beta(E', E) \cap \operatorname{Ker} A^{\perp}$ on $A'(F') = \operatorname{Ker} A^{\perp}$.*

Proof. Let $A = J\breve{A}K$ be the canonical decomposition of weak homomorphism with the weakly closed range. Since the adjoint to a weak continuous mapping is strongly continuous, $A' = K'\breve{A}'J'$ is strongly continuous and, due to the above, this decomposition is the canonical decomposition of the operator A'. Therefore, $\breve{A}' : (F'/\operatorname{Ker} A', \beta(F', F)/\operatorname{Ker} A') \to (A'(F'), \beta(E', E) \cap A'(F'))$ is injective and continuous mapping. On the other hand,

$$\breve{A}' : (F'/\operatorname{Ker} A', \beta(F'/\operatorname{Ker} A', A(F)) \to (A'(F'), \beta(A'(F'), E/\operatorname{Ker} A))$$

is a strong isomorphism, since $\breve{A}'^{(-1)} = (\breve{A}^{-1})'$.

From here, taking into account the known properties of strong topologies, we obtain that the following diagram is valid:

$$\begin{tikzcd}
(F'/\operatorname{Ker} A', \beta(F', F)/\operatorname{Ker} A') \arrow[r] \arrow[d] & (A'(F'),\ \beta(E', E) \cap A'(F')) \\
(F'/\operatorname{Ker} A', \beta(F'/\operatorname{Ker} A', A(E)) \arrow[r, shift left] & (A'(F'),\ \beta(A'(F'), E/\operatorname{Ker} A)) \arrow[l, shift left] \arrow[u]
\end{tikzcd}$$

where the vertical arrows indicate the identity continuous algebraic isomorphisms. From this it is not difficult to obtain the proof of Theorems 2.7.8. $\square$

Corollary 1. *Let $A : E \to F$ be a homomorphism of the (DF)-space $(E, \mathfrak{T})$ in an arbitrary LCS $(F, \mathfrak{T})$. Then A' is strong homomorphism.*

Indeed, by virtue of Theorem 1 ([82], p. 401), we have that $(E/\operatorname{Ker} A, \mathfrak{T}_1/\operatorname{Ker} A)$ is a (DF)-space and therefore $((E/\operatorname{Ker} A)', \beta(\operatorname{Ker} A^{\perp}, \ E/\operatorname{Ker} A)) = (\operatorname{Ker} A^{\perp}, \beta(E', E) \cap \ Ker A^{\perp}))$. Further, from the condition we obtain that $(A(E), \mathfrak{T}_2 \cap A(E))$ is a (DF)-space and therefore, by virtue of Theorem 2 ([82], p. 401), we obtain the equality

$$((A(E)', \beta(F'/\operatorname{Ker} A', A(E))) = (F'/\operatorname{Ker} A', \beta(F', F)/\operatorname{Ker} A').$$

Consequently, it follows from Theorem 2.7.8 that A' is a strong homomorphism.

It should be noted that Corollary 1 of Theorem 2.7.8 is valid for the space $(E, \mathfrak{T}_1)$ of type (DFS), in particular, for Vladimirov algebra [178, 188]. Moreover, if the space $(F, \mathfrak{T}_2)$ is also a space of type (DFS), then the converse statement is also true ([43], p. 105), i.e. A is a homomorphism if A is continuous and A' is a strong homomorphism. In Section 2.7.4, Example 3 will be given that the last statement is not true for arbitrary (DF)-spaces.

Corollary 2. *Let $A : E \to F$ be a weak homomorphism of the Fréchet space E in the Fréchet space F. The adjoint operator A' is a strong homomorphism if and only if $\beta((E/\operatorname{Ker} A)', E\,/\operatorname{Ker} A)) = \beta(E', E) \cap \operatorname{Ker} A^{\perp}$ to $(E/\operatorname{Ker} A)' = \operatorname{Ker} A^{\perp}$ and $\beta(A(E)',\ A(E)) = \beta(F', F)/\operatorname{Ker} A'$ on $A(E)' = F'/\operatorname{Ker} A'$.*

From Theorem 12 [65] it follows that the Fréchet–Schwarz spaces E and F satisfy the conditions of Corollary 2. Moreover, similarly to the proof of Corollary 1, it can be proved that if A is a homomorphism of the space (FS) in an arbitrary LCS, then A' is a strong homomorphism. We will indicate here another class of Fréchet spaces, containing non-reflexive spaces, satisfying the conditions of Corollary 2.

Proposition 2.7.9. *Let A be a homomorphism of the quejection $B \times \omega$ in the LCS $(F, \mathfrak{T}_2)$, where B is the Banach space and ω is the space of all numerical sequences.Then the adjoint mapping A' is a strong homomorphism. Furthermore, if the space $(F, \mathfrak{T}_2)$ is a Fréchet space, then the adjoint mapping of arbitrary order is a strong homomorphism.*

Proof. Let the conditions of Corollary 2 of Theorem 2.7.8 are fulfilled. Indeed, as is known, the quotient space $B \times \omega$ is isomorphic either to the Banach space B_1 or to the space $B_2 \times \omega$, where B_2 is also a Banach space. Therefore, the space $((B \times \omega)/\operatorname{Ker} A)'$, $\beta((B \times \omega)/\operatorname{Ker} A)'$, $(B \times \omega)/\operatorname{Ker} A))$ is a strictly (LB)-space. We prove that the latter space is isomorphic to the space $(\operatorname{Ker} A^{\perp}, \beta(B' \times \varphi, B \times \omega) \cap \operatorname{Ker} A^{\perp})$, where $\varphi = \omega'$ is the space of all finite sequences. Indeed, since $\operatorname{Ker} A$ is a quojection, we have the equality $((\operatorname{Ker} A)', \beta((\operatorname{Ker} A)',\ \operatorname{Ker} A)) = ((B' \times \varphi)/\operatorname{Ker} A^{\perp}, \beta(B' \times \varphi, B \times \omega)/\operatorname{Ker} A^{\perp})$, where the latter is a strict (LB)-space. Then, by virtue of the first part of Theorem 2 from [142], it turns out that the space $(\operatorname{Ker} A^{\perp}, \beta(B' \times \varphi, B \times \omega) \cap \operatorname{Ker} A^{\perp})$ is also a strict (LB)-space. By virtue of the well-known theorem about openness of a continuous mapping of strict (LF)-spaces from [43], we obtain the coincidence of the above-mentioned two topologies on $\operatorname{Ker} A^{\perp}$.

By condition, the quotient space $(B \times \omega)/\operatorname{Ker} A$ is isomorphic to the space $(A(B \times \omega), \mathfrak{T}_2 \cap A(B \times \omega))$ and, therefore, the latter space is distinguished. Further, applying Theorem 8 from [65], we obtain the equality

$$\big((A(B\times\omega))', \beta((A(B\times\omega))', A(B\times\omega))\big) = (F'/\operatorname{Ker} A', \beta(F',F)/\operatorname{Ker} A'\big).$$

Therefore, A' is a strong homomorphism. The second adjoint A'' to the homomorphism A, i.e. the adjoint to the mapping $A' : (F', \beta(F', F)) \to (E', \beta(E', E))$, is a strong homomorphism of the space $(E'', \beta(E'', E'))$ in $(F'', \beta(F'', F'))$, by using Corollary 1 of Theorem 2.7.8 for the homomorphism A'. The remaining part of the statement follows from the obvious fact that the strong bidual to the space $B\times\omega$ is identified with the space $B''\times\omega$. □

It should be noted that, as follows from the proof of Propositions 2.7.9, every weakly closed subspace of the strongly dual space to the space $B\times\omega$ is a strict (LB)-space. By a similar reasoning, one can prove that each closed subspace of the space $B\times\varphi$ is a strict (LB)-space.

It should also be noted that Proposition 2.7.9 does not hold for an arbitrary quojection. Indeed, in [65], it is constructed an example of a canonical homomorphism of a Fréchet-Montel space on a Banach space, whose adjoint is not a strong homomorphism. It also states that using this example, one can construct a canonical homomorphism k of the quojection $(l^p)^N$ $(1\le p<\infty)$ on its quotient space such that the adjoint mapping k' will not be again a strong homomorphism. Furthermore, it is indicated that an analogous example of the canonical homomorphism can be constructed for the quojection $C]\,0,1[$ of the space of continuous functions on $]\,0,1[$, which is equipped with a compact convergence topology. In there both cases the weakly closed subspace $\operatorname{Ker} k^{\perp}$ in the strong topologies $\beta(\operatorname{Ker} k^{\perp}, (l^p)^N/\operatorname{Ker} k))$ and $\beta(\operatorname{Ker} k^{\perp}, C]\,0,1[/\operatorname{Ker} k)$ is a strict (LB)-space, i.e. two of the three mentioned in [65] topologies coincide. In the topology induced from the strong dual strict (LB)-space, the subspace $\operatorname{Ker} k^{\perp}$ is not even (DF)-space, although it has a fundamental sequence of bounded sets. It should be specially noted that unlike the above-mentioned canonical homomorphism, the quotient spaces $(l^p)^N/\operatorname{Ker} k$ and $C]\,0,1[\,/\operatorname{Ker} k$ cannot be Banach one by virtue of the following Proposition.

Proposition 2.7.10. *Let $k : E \to F$ be a canonical homomorphism of the prequojection E to a Banach space $(F, \|\cdot\|)$. Then the adjoint mapping k' is a strong homomorphism.*

Proof. Indeed, the adjoint mapping k' is a strongly continuous and injective mapping of the Banach space $(F', \|\cdot\|')$ in the strict (LB)-space $(E', \beta(E', E)) = s\cdot\lim\limits_{\rightarrow} F_n$ with a weak and hence strongly closed image $k'(F') = \operatorname{Ker} k^{\perp}$. By Theorem 4 from ([82], p. 285), we have that k' is a continuous mapping of the Banach space F' in some Banach space $F_{n_\circ}$, where $k'(F')$ is closed. Hence it follows that the strong topology on $\operatorname{Ker} k^\perp$ coincides with the induced topology from F_{n_0} and therefore from $(E', \beta(E', E))$, i.e. k' is a strong monomorphism. $\square$

The answer to the following question is known: is the quojection E isomorphic to the space $B \times \omega$ if the adjoint to an arbitrary homomorphism of the space E to an LCS F is a strong homomorphism?

We do not known whether in Proposition 2.7.9 the space ω can be replaced by an arbitrary strict Fréchet-Schwartz space.

In the work [119] (see also [188], p. 105), in terms of the duality functor D, the necessary and sufficient conditions were found if: a) the strong dual to subspace was identified with the quotient space of the strong dual space and, b) the strong dual to the quotient space was identified with the subspace of the strong dual. Using these results in combination with Theorem 2.7.8, the following proposition is obtained.

Proposition 2.7.11. *Let A be a homomorphism of the LCS E into the LCS F. The adjoint mapping A' is a strong homomorphism if and only if $D^1_M(\operatorname{Ker} A) = 0$ and $D^+_M(A(E)) = 0$, where M is an arbitrary set of a sufficiently large cardinality, D^1_M is the first derivative of the duality functor D, and D^+_M is an additional derivative.*

The stability of behavior of the class homomorphism in the case of (covariant) functorial topologies, i.e. associated quasi barreled, ultra bornological, nuclear topology are discussed in the work [40]. Homological algebra and derived functors are also utilized in [35] to investigate when the adjoint (transpose) operator of homomorphism in the category of locally convex spaces is again a homomorphism.

It would be interesting to find similar relations (connections) for second adjoint mappings (betransposes) and other topologies to be considered below.

Theorem 2.7.12. *Let A be a weak homomorphism of the LCS E in the LCS F with a weakly closed image. Then the adjoint mapping A' is a $\mathfrak{T}_k$-homomorphism, i.e. a homomorphism of the space $(F', \tau(F', F))$ in $(E', \tau(E', E))$ if and only if*

$$((E/\operatorname{Ker} A)', \tau((E/\operatorname{Ker} A)', E/\operatorname{Ker} A)) = (\operatorname{Ker} A^\perp, \tau(E', E) \cap \operatorname{Ker} A^\perp).$$

This theorem follows from Theorem 2.7.7 due to the well-known equality

$$\big(A(E)', \tau(A(E)', A(E))\big) = (F'/\operatorname{Ker} A', \tau(F', F)/\operatorname{Ker} A'),$$

which is obtained from Theorem 4 ([82], p. 278).

Theorem 2.7.13. *Let A be a weak homomorphism of the LCS E in the LCS F with a closed image. The adjoint mapping A' is $\mathfrak{T}_c$-homomorphism, i.e. homomorphism of the space $(F', \mathfrak{T}_c(F))$ in the space $(E', \mathfrak{T}_c(E))$ if and only if* $((E/\operatorname{Ker} A)', \mathfrak{T}_c(E/\operatorname{Ker} A)) = (\operatorname{Ker} A^{\perp}, \mathfrak{T}_c(E) \cap \operatorname{Ker} A^{\perp})$ *and* $(A(E)', \mathfrak{T}_c(A(E)) = (F'/\operatorname{Ker} A', \mathfrak{T}_c(F)/\operatorname{Ker} A')$.

Corollary 1. *Let A be a homomorphism of the Fréchet space E into a quasicomplete LCS F, then the adjoint mapping A' is a $\mathfrak{T}_c$-homomorphism.*

This corollary follows from Theorems 5 and 6 ([82], p. 278) and Theorem 2.7.13.

Corollary 2. *Let A be a homomorphism of Montel (DF)-space E into a quasicomplete LCS F, then the adjoint mapping A' is a $\mathfrak{T}_c$-homomorphism.*

Corollary 3. *Let A be a monomorphism of the LCS E into a quasicomplete LCS F, then the adjoint mapping A' is $\mathfrak{T}_c$-homomorphism.*

We will now study this problem by endowing the adjoint spaces by inductive topologies. Inductive topology $\mathfrak{T}_I(E)$ was introduced in [137] as the strongest locally convex topology on E', preserving equicontinuous sets by bounded. This topology coincides with the strong topology, in particular, for quasi-normed spaces and differs from it in the case of nondistinguished Fréchet spaces.

Theorem 2.7.14. *Let A be a homomorphism of the LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$. The adjoint operator A' is $\mathfrak{T}_I$-homomorphism, i.e. homomorphism of the space $(F', \mathfrak{T}_I(F))$ into $(E', \mathfrak{T}_I(E))$ if and only if $((E/\operatorname{Ker} A)', \mathfrak{T}_I(E/\operatorname{Ker} A)) = (\operatorname{Ker} A^{\perp}, \mathfrak{T}_I(E) \cap \operatorname{Ker} A^{1})$.*

This theorem does not follow from the general Theorem 2.7.7, but it can be proved by similar reasoning. Indeed, due to [141], we have that

$$\big(A(E)', \mathfrak{T}_I(A(E))\big) = (F'/\operatorname{Ker} A', \mathfrak{T}_I(F)/\operatorname{Ker} A').$$

Further, A' is $\mathfrak{T}_I$-continuous. It is enough to prove that it is continuous the restriction of the mapping $A' : (F', \mathfrak{T}_I(F)) \to (E', \mathfrak{T}_I(E))$ on a Banach space E'_{V° spanned by V°, where V is $\mathfrak{T}_2$-neighborhood in F. Let W be $\mathfrak{T}_I(E)$-neighborhood of the space $(E', \mathfrak{T}_I(E))$ and $U = A^{-1}(V)$ be a neighborhood in E. By definition of the topology $\mathfrak{T}_I(E)$, there is $\lambda > 0$ such that $\lambda U^0 \subset W$. On the other hand, passing in polars in the equality $AU = V$ we get that $V^0 = (A(U))^0 = A'^{(-1)}(U^0)$, i.e. $U^0 = A'(V^0)$. That's why, $A'(\lambda V^\circ) = \lambda U^\circ \subset W$.

To complete the proof of Theorem 2.7.14, we have also take into account the inequality $\mathfrak{T}_I(E) \cap \operatorname{Ker} A^{\perp} \leq \mathfrak{T}_I(E/\operatorname{Ker} A)$ between two topologies on $\operatorname{Ker} A^{\perp}$. Indeed, let W be $\mathfrak{T}_I(E) \cap \operatorname{Ker} A^{\perp}$-neighborhood in $\operatorname{Ker} A^{\perp}$, then it has the form

$W = V \cap \operatorname{Ker} A^{\perp}$, where V is $\mathfrak{T}_I(E)$-neighborhood, i.e. V absorbs equicontinuous sets U^0 of the space E'. Then W absorbs every set of the form $U^0 \cap \operatorname{Ker} A^{\perp}$. It follows that W absorbs the sets of the form $k(U)^{\circ}$, where $k : E \to E/\operatorname{Ker} A$ is a canonical mapping, since the equalities $k(U)^0 = k'^{(-1)}(U^0) = U^0 \cap \operatorname{Ker} A^{\perp}$ are true. But this means that W is a $\mathfrak{T}_I(E/\operatorname{Ker} A)$-neighborhood in $\operatorname{Ker} A^{\perp}$.

In [141] (see also [119]), the sufficient conditions were obtained in order that the adjoint of a homomorphism is a $\mathfrak{T}_I$-homomorphism. From these results and Theorem 2.7.14 it follows that if for a homomorphism A, the kernel $\operatorname{Ker} A$ is a quasi-normed metrizable LCS, then $((E/\operatorname{Ker} A)', \ \mathfrak{T}_I(E/\operatorname{Ker} A)) = (\operatorname{Ker} A^{\perp}, \mathfrak{T}_I(E) \cap \operatorname{Ker} A^{\perp})$ and therefore A' is $\mathfrak{T}_I$-homomorphism.

2.7.3 Second adjoint operator to a homomorphism between locally convex spaces

Let $(E, \mathfrak{T})$ be an LCS, E' be its dual space, and E'' be its second dual space, i.e. $E'' = (E', \beta(E', E))'$. For weakly continuous linear mapping A of the LCS$(E, \mathfrak{T}_1)$ into the LCS $(F, \mathfrak{T}_2)$, it is defined the second dual mapping A'' on E'' by the equality

$$\langle A''x'', \, y'\rangle = \langle x', \, A'y'\rangle,$$

assuming that it holds for all $x'' \in E''$ and $y' \in F'$. Obviously, the restriction of A'' to E coincides with A. Due to the fact that A'' is the continuous mapping of the space $(E'', \sigma(E'', E'))$ in the space $(F'', \sigma(F'', F'))$ and E is $\sigma(E'', E')$ dense in E'', it turns out that A'' is the extension of A to E'', by continuity in the topology $\sigma(E'', E')$. Moreover, through $\sigma(E'', E')$ (resp. $\beta(E'', E')$) it is denoted the weak (resp. strong) topology of the dual pair $\langle E', E''\rangle$ on E''. Through $\mathfrak{T}_n(E')$ it is denoted the natural topology on E'', i.e. the topology of uniform convergence on equicontinuous sets of the adjoint space E'.

In this section, we study the second adjoint mapping to a homomorphism in the mentioned topologies of second dual spaces. The conditions are obtained under which the second adjoint is again a homomorphism, regardless of whether the adjoint mapping is a strong homomorphism. Let us first give simple statements about the second adjoint to a weak homomorphism, which are derived from the known results.

Proposition 2.7.15. *Let A be a weakly continuous map LCS $(E, \, \mathfrak{T}_1)$ in LCS $(F, \mathfrak{T}_2)$. Then the following statements are valid:*

a) *The second adjoint mapping A'' is a weak homomorphism, i.e. a homomorphism of the space $(E'', \sigma(E'', E'))$ in the space $(F'', \, \sigma(F'', F'))$ if and only if the image $A'(F')$ is $\sigma(E', E'')$ closed in $(E', \, \sigma(E', \, E''))$, where $\sigma(E', E'')$ is the weak topology of the dual pairs $\langle E', \, E''\rangle$ on E'.*

b) *If A is a weak homomorphism of the space E into the space F, then A'' is a weak homomorphism of the space $(E'', \sigma(E'', E'))$ in the space $(F'', \sigma(F'', F'))$. Further, there is an example of a weakly continuous mapping that is not weakly open, and the second adjoint map is a weak homomorphism.*

c) *An image $A''(E'')$ is $\sigma(F'', F')$ closed if and only if A' is a weak homomorphism of the space $(F', \sigma(F', F''))$ in the space $(E', \sigma(E', E''))$.*

d) *If A is a weak homomorphism of the space E into the space F whose adjoint map is strong homomorphism, then A'' is a weak homomorphism of the space $(E'', \sigma(E'', E'))$ in the space $(F'', \sigma(F'', F'))$ with a closed image. Moreover, the following equalities hold: $A(E)'' = (\operatorname{Ker} A')^{\perp} = A(E)^{\perp\perp} = A''(E'')$ and $(E/\operatorname{Ker} A)'' = (\operatorname{Ker} A^{\perp})' = E''/\operatorname{Ker} A'' = (A'(F'))'$.*

Proof. a) The adjoint mapping A' to the weakly continuous mapping A is strongly continuous, i.e. is continuous in some topologies which are compatible with the dual systems $\langle F', F'' \rangle$ and $\langle E', E'' \rangle$. Therefore, A' is a continuous mapping of the space $(F', \sigma(F', F''))$ in the space $(E', \sigma(E', E''))$ and its adjoint mapping A'' is weakly continuous. To complete the proof of statement a), it is enough to apply Theorem 2 from ([83], p. 5) in the case of the dual pairs $\langle E'', E' \rangle$ and $\langle F'', F' \rangle$ and the mapping A''.

b) From the condition it turns out that A' is the weakly continuous mapping of the space $(F', \sigma(F', F''))$ into the space $(E', \sigma(E', E''))$ with a $\sigma(E', E)$ closed image of $A'(F')$. But since on E' the inequality $\sigma(E', E) \leq \sigma(E', E'')$ holds, then $A'(F')$ is $\sigma(E', E'')$ closed in E'. By statement a), A'' is a weak homomorphism.

In Section 2.7.4, there will be an example of the weakly continuous mapping k_2 (see Example 3), whose second adjoint k_2'' is a weak homomorphism, but k_2 is not weakly open. This shows that it is not fair the converse of statement b).

c) It was proved above that A' is a continuous mapping of the space $(F', \sigma(F', F''))$ in $(E', \sigma(E', E''))$. It remains to apply Theorem 2 from ([83], p. 5) for dual pairs $\langle F', F'' \rangle$ and $\langle E', E'' \rangle$ and mapping A'.

d) By assumption, A' is a strong homomorphism, i.e. A' is a homomorphism in some topologies compatible with the dualities $\langle F', F'' \rangle$ and $\langle E', E'' \rangle$. Therefore, A'' is the homomorphism of the space $(E'', \sigma(E'', E'))$ in the space $(F'', \sigma(F'', F'))$ with a closed image. Further, by definition and by Theorem 2.7.8, we have that

$$\begin{aligned} A(E)'' &= \big(A(E)', \beta(A(E)', A(E))\big)' = (F'/\operatorname{Ker} A', \beta(F', F)/\operatorname{Ker} A')' \\ &= (\operatorname{Ker} A')^{\perp} = A(E)^{\perp\perp}, \end{aligned}$$

where $\underline{A(E)^{\perp\perp}}$ is $\sigma(F'', F')$-closure of $A(E)$ in F''. Since $\overline{A''(E'')} = (\operatorname{Ker} A')^{\perp}$, where $\overline{A''(E'')}$ means $\sigma(F'', F')$-closure of the set $A''(E'')$ and $A''(E'')$ is weakly closed, we have $A''(E'') = (\operatorname{Ker} A')^{\perp}$. $\square$

Theorem 2.7.16. *Let A be a weak homomorphism of the LCS E into the LCS F with the closed image whose adjoint mapping A' is a strong homomorphism. Then the following statements are equivalent:*

a) *The second conjugate mapping A'' is strong homomorphism of the space $(E'', \beta(E'', E'))$ in the space $(F'', \beta((F'', F'))$.*

b) *The equalities $\beta((A'(F')', A'(F')) = \beta(E'', E')/\operatorname{Ker} A''$ on $(A'(F')' = E''/\operatorname{Ker} A''$ and $\beta((F'/\operatorname{Ker} A')', \ F'/\operatorname{Ker} A') = \beta(F'', F') \cap A''(F'')$ on $(E'/\operatorname{Ker} A')' = A''(F'')$ are valid, where $A'(F')$ is considered in the induced topology $\beta(E', E) \cap A'(F')$, and the quotient space $F'/\operatorname{Ker} A'$ is considered in the quotient topology.*

c) *The following equalities are valid: $\beta((E/\operatorname{Ker} A)'', (E/\operatorname{Ker} A)') = \beta(E'', E')/\operatorname{Ker} A''$ to $(E/\operatorname{Ker} A)'' = E''/\operatorname{Ker} A''$ and $\beta(A(E)'', A(E)') = \beta(F'', F') \cap A''(F'')$ to $A(E)'' = A''(E'')$, where $\beta((E/\operatorname{Ker} A)'', \ (E/\operatorname{Ker} A)')$ is the strong topology of the space $(E/\operatorname{Ker} A)''$ and $\beta(A(E)'', A(E)')$ is the strong topology of the space $A(E)''$.*

Proof. a) $\Longleftrightarrow$ b) follows from Theorem 2.7.1, applying it to the strong homomorphism A' of the space $(F', \beta(F', F))$ in the space $(E', \beta(E', E))$.

a) $\Longrightarrow$ c). Let A'' be the strong homomorphism, then by virtue of statement b) and by the definition of a strong second dual space, we have

$$\begin{aligned}\big(A(E)'', \beta(A(E)'', A(E)')\big) &= \big((F'/\operatorname{Ker} A')', \beta((F'/\operatorname{Ker} A')', F'/\operatorname{Ker} A'))\big)\\ &= ((\operatorname{Ker} A')^{\perp}, \beta(F'', F') \cap (\operatorname{Ker} A')^{\perp}) = (A''(E''), \beta(F'', F') \cap A''(E''))\end{aligned}$$

and

$$\begin{aligned}&\big((E/\operatorname{Ker} A)'', \beta((E/\operatorname{Ker} A)'', (E/\operatorname{Ker} A)')\big)\\ &\qquad\qquad = ((\operatorname{Ker} A^{\perp})', \beta((\operatorname{Ker} A^{\perp})', \operatorname{Ker} A^{\perp}))\\ &= \big(((A'(F'))', \beta((A'(F'))', A'(F'))\big) = (E''/\operatorname{Ker} A'', \beta(E'', E')/\operatorname{Ker} A'').\end{aligned}$$

c) $\Longrightarrow$ a). Since A is the weak homomorphism, then the spaces $\big((E/\operatorname{Ker} A)'', \beta((E/\operatorname{Ker} A)'', (E/\operatorname{Ker} A)')\big)$ and $\big(A(E)'', \beta(A(E)'', A(E)')\big)$ are isomorphic. From here, by condition, we obtain that $(E''/\operatorname{Ker} A'', \beta(E'', \ E')/\operatorname{Ker} A'')$ and $(A(E)'', \beta(A(E)'', A(E)')$ are isomorphic, i.e. A'' is a strong homomorphism. $\square$

It should be noted that when studying the second conjugate to the homomorphism the following diagram can be applied:

$$\begin{tikzcd}
 & ((E/\operatorname{Ker} A)'',\ \beta((E/\operatorname{Ker} A)'',\ (E/\operatorname{Ker} A)')) \arrow[r, "A_5"] & (A(E)'',\ \beta(A(E)'',\ A(E)')) \arrow[d, "A_7"] \\
(E'',\ \beta(E'',E')) \arrow[ur] \arrow[r] \arrow[dr] & (E''/\operatorname{Ker} A'', \beta(E'',E')/\operatorname{Ker} A'') \arrow[u, "A_1"] \arrow[r, "A_4"] \arrow[dr, "A_9"] & (A''(E''), \beta(F'',F') \cap A''(E'')) \arrow[ul, "A_2"] \arrow[dl, "A_3"'] \\
 & ((A'(F'), \beta(E',E) \cap A'(F'))'_\beta \arrow[r, "A_6"] & ((F'/\operatorname{Ker} A', \beta(F',F)/\operatorname{Ker} A')'_\beta \arrow[u, "A_8"']
\end{tikzcd}$$

where A_1–A_9 denotes continuous identity mappings. In particular, under the a priori conditions of Theorem 2.7.16 one can prove that all mappings A_1–A_9 are continuous algebraic isomorphisms and the mappings A_1, A_5, A_6 and A_9 are topological isomorphisms. Theorem 2.7.16 states that A_4 is a topological isomorphism if and only if the operators A_3 and A_8, respectively A_2 and A_7, are topological isomorphisms. This implies, in particular, that A_4 may not be topological isomorphism when A_5 and A_6 are topological isomorphisms.

Corollary 1. *Let k be a homomorphism of the (DF)-space E on the (DF)-space F such that $(k'(F'),\ \beta(E',E) \cap k'(F'))$ is a distinguished Fréchet space. Then k'' is the strong homomorphism $(E'',\ \beta(E'',E'))$ on $(F'', \beta(F'',F'))$.*

Corollary 2. *Let A be a homomorphism of the Fréchet space E in the Fréchet space F such that A' is strong homomorphism. Then A'' is a strong homomorphism.*

The validity of this statement follows from the fact that all spaces mentioned in the diagram are Fréchet spaces. It is not known whether Corollary 2 holds without the requirement that A' is strongly homomorphic.

The linear operator $A : (E, \mathfrak{T}_1) \to (F, \mathfrak{T}_2)$ is called an almost open mapping of LCS $(E, \mathfrak{T}_1)$ in LCS $(F, \mathfrak{T}_2)$ if the $\mathfrak{T}_2$-closure of the image $\overline{A(U)}$ of each $\mathfrak{T}_1$-neighborhood U is $\mathfrak{T}_2$-neighborhood in $\overline{A(E)}$.

Theorem 2.7.17. *Let A be continuous and almost open operator of LCS $(E, \mathfrak{T}_1)$ in LCS $(F, \mathfrak{T}_2)$. Then the second adjoint mapping A'' is the almost weakly open in natural topologies, i.e. $\sigma(F'',\ F')$-closure of the A'' image of each $\mathfrak{T}_n(E')$-neighborhood of the space E'' is $\mathfrak{T}_n(F')$-neighborhood in F''.*

Proof. Let us prove that for each $\mathfrak{T}_1$-neighborhood of U, the $\sigma(F'', F')$-closure of the A''-image $U^{\circ\circ} = \overline{U}^{\sigma(E'',E')}$ is a $\mathfrak{T}_n(\sigma(F'', F')$-neighborhood in F''. By virtue of Theorem 4 ([83], p. 25), the equality $A'(\mathfrak{M}_2) = \mathfrak{M}_1 \cap A'(F')$ is valid, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous sets in E' and F'. Therefore, for U° there is $M \in \mathfrak{M}_2,\ M \subset V^\circ$, where V is $\mathfrak{T}_2$-neighborhood such that $A'(V^0) \supset$

$A'(M) = U^\circ \cap A'(F')$. Since A' is a strongly continuous mapping, A' is continuous mapping of the space $(F', \sigma(F', F''))$ in the space $(E', \sigma(E', E''))$. Passing in polars in E'' in the above inclusion, we get that $(A'(V^\circ))^\circ = A''^{(-1)}(V^{\circ\circ}) \subset \overline{U^{\circ\circ} + A'(F')^\perp}^{\sigma(E'',E')}$, where the latter means $\sigma(E'', E')$-closure of the set $U^{\circ\circ} + A'(F')^\perp$. It follows from this that

$$V^{\circ\circ} \cap A''(E'') = A''\overline{(U^{\circ\circ} + \operatorname{Ker} A'')}^{\sigma(F'',F')} \subset \overline{A''(U^{\circ\circ} + \operatorname{Ker} A'')} = \overline{A''(U^{\circ\circ})},$$

i.e. A'' is almost $\sigma(F'', F')$-open in natural topologies. $\square$

It should be noted that if under the conditions of Theorem 2.7.17 A was even homomorphism (A will be such when E is a Ptak space, i.e. B-complete), then, generally speaking, it is impossible assert more than in Theorem 2.7.17. However, we present the particular case when this is possible.

Corollary 1. *Let A be a continuous and almost open mapping of LCS $(E, \mathfrak{T}_1)$ into LCS $(F, \mathfrak{T}_2)$. Further, let for any $\mathfrak{T}_1$-neighborhood U, the inclusion $A''\overline{(U^{00} + \operatorname{Ker} A'')}^{\sigma(E'',E')} \subset n_U A''(U^{00})$ be true, where U^{00} is the bipolar of the set U in E'' and $n_U \in \mathbb{N}$. Then A'' is a homomorphism in natural topologies, i.e. A'' is a homomorphism of the space $(E'', \mathfrak{T}_n(E'))$ in the space $(F'', \mathfrak{T}_n(F'))$.*

Proof. From the proof of Theorem 2.7.17 and the conditions, it turns out that for every $\mathfrak{T}_1$-neighborhood U, there is a $\mathfrak{T}_2$-neighborhood V, for which the following inclusion holds: $V^{\circ\circ} \cap A''(E'') \subset n_U A''(U^{\circ\circ})$. $\square$

Corollary 2. *Let A be a continuous and almost open mapping of LCS $(E, \mathfrak{T}_1)$ to LCS $(F, \mathfrak{T}_2)$ and natural topology $\mathfrak{T}_n(E')$ of the space E'' is compatible with the duality $\langle E'', E' \rangle$, then A'' is a homomorphism in natural topologies.*

Indeed, in this case the $\sigma(E'', E')$-closure of the set $U^{\circ\circ} + \operatorname{Ker} A''$ coincides with the closure in the natural topology and therefore $\overline{(U^{\circ\circ} + \operatorname{Ker} A'')}^{\sigma(E'',E')} \subset \operatorname{Ker} A'' + 2U^{\circ\circ}$, i.e. $A''\overline{(U^{\circ\circ} + \operatorname{Ker} A'')}^{\sigma(E'',E')} \subset 2A''(U^{\circ\circ})$.

This result generalizes Theorem 4 from ([83], p. 11). It should be noted that if the spaces $(E, \mathfrak{T}_1)$ and $(F, \mathfrak{T}_2)$ are quasi-barreled, then natural topologies coincide with strong topologies on E'' and F''. Therefore, in this case, in Corollaries 1 and 2 we can talk about strong homomorphisms.

Proposition 2.7.18. *Let A be a homomorphism of the LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$ such that $(E/\operatorname{Ker} A)'' = E''/\operatorname{Ker} A''$. Then the second adjoint mapping A'' is a homomorphism in natural topologies if and only if*

$$((E/\operatorname{Ker} A)'', \mathfrak{T}_n(\operatorname{Ker} A^\perp)) = (E''/\operatorname{Ker} A'', \mathfrak{T}_n(E')/\operatorname{Ker} A'').$$

It is known that the second adjoint to the continuous mapping is continuous in natural topologies. Next, for each neighborhood U of the space E, the inclusion $k''(U^{00}) \subset k(U)^{00}$ holds, where $k : E \to E/\operatorname{Ker} A$ is the canonical map. Therefore, between two topologies on $(E/\operatorname{Ker} A)'' = E''/\operatorname{Ker} A''$ the inequality $\mathfrak{T}_n(E')/\operatorname{Ker} A'' \geq \mathfrak{T}_n(\operatorname{Ker} A^{\perp})$ is true. From the homomorphism of A and from the condition it follows that $A(E)'' = A''(E'')$. It is also easy to prove that the topologies on these spaces $\mathfrak{T}_n(F'/\operatorname{Ker} A')$ and $\mathfrak{T}_n(F') \cap A''(F'')$ coincide. So, due to the above said, the following diagram holds:

$$\begin{tikzcd}
 & ((E/\operatorname{Ker} A)'', \mathfrak{T}_n(E/\operatorname{Ker} A)') \arrow[r, shift left] & (A(E)'', \mathfrak{T}_n(A(E)')) \arrow[l, shift left] \arrow[d, shift left] \arrow[dr] & \\
(E'', \mathfrak{T}_n(E')) \arrow[ur] \arrow[dr] & & & (F'', \mathfrak{T}_n(F')) \\
 & (E''/\operatorname{Ker} A'', \mathfrak{T}_n(E')/\operatorname{Ker} A'') \arrow[uu] \arrow[r] & (A''(E''), \mathfrak{T}_n(F') \cap A''(E'')) \arrow[u, shift left] \arrow[ur] &
\end{tikzcd}$$

where the vertical and horizontal arrows indicate the above mentioned mappings. From here our statement follows similar to the above reasoning. □

It should be noted that there is an example of a Fréchet space E and of the canonical homomorphism k: $E \to E/\operatorname{Ker} k$ for which $(E/\operatorname{Ker} k)'' \supset E''/\operatorname{Ker} k''$ and this inclusion is proper. Indeed, let $E = (l^1)^N$ and F be a closed subspace of E such that on $F^{\perp}$ two strong topologies give different dual spaces [65]. We then have that if k: $E \to E/F$, then

$$\begin{aligned}(k(E))'' = (E/F)'' = ((E/F)', \beta((E/F)', E/F))' \not\supseteq (F^{\perp}, \beta(E', E) \cap F^{\perp})' \\ = E''/\operatorname{Ker} k''.\end{aligned}$$

It is not known whether in this example the second conjugate mapping k'' would be a strong homomorphism.

2.7.4 Notes on some homomorphisms of Fréchet spaces, (DF)-spaces, strict (LF)-spaces and their adjoint mappings

Example 1. It is well known the sufficiently complicated example of Fréchet–Montel space $(E, \mathfrak{T})$ from [65] (see also [83], p. 22), a quotient space of which is isomorphic to the Banach space l^1. Let $k : E \to l^1$ be a canonical homomorphism, i.e. the quotient topology $\mathfrak{T}/\operatorname{Ker} k$ coincides with the topology of the norm of the space l^1. It is known that k is strong homomorphism, i.e. $\beta(E, E')/\operatorname{Ker} k = \beta(l^1, l^\infty)$. By virtue of results of Theorem 2.7.1, we find that

k is also $\mathfrak{T}_{b^*}$-homomorphism (corollary of Theorem 2.7.3), $\mathfrak{T}_k$-homomorphism (Corollary 1 of Theorem 2.7.4) and $\mathfrak{T}^\times$-homomorphism. However, k is not a $\mathfrak{T}_c$-homomorphism, since $\mathfrak{T}_c(E')/\operatorname{Ker} k = \beta(E,E')/\operatorname{Ker} k = \beta(l^1, l^\infty) \neq \mathfrak{T}_c(l^\infty)$. It is known [65] that the conjugate mapping k' is not a strong monomorphism, therefore,

$$((E/\operatorname{Ker} k)',\ \beta((E/\operatorname{Ker} k)', E/\operatorname{Ker} k)) = (l^\infty, \beta(l^\infty, l^1))$$
$$\neq (k'(l^\infty), \beta(E',E) \cap k'(l^\infty)).$$

Further, as is known, the topology $\beta(E',E) = \mathfrak{T}_c(E')$ induces on $k'(l^\infty) = \operatorname{Ker} k^\perp$ the topology of uniform convergence on all relatively compact subsets $E/\operatorname{Ker} k = l^1$. From here it also turns out that $(l^\infty, \mathfrak{T}_c(l^1)) = (k'(l^\infty), \mathfrak{T}_c(E) \cap k'(l^\infty))$, i.e. k' is a $\mathfrak{T}_c$-monomorphism by Corollary 1 of Theorem 2.7.13. From Theorem 2.7.14 it follows that k' is not $\mathfrak{T}_I$-monomorphism, since in topology $\mathfrak{T}_I(E) \cap k'(l^\infty) = \mathfrak{T}_c(E) \cap k'(l^\infty)$ the space $k'(l^\infty)$ is not bornological space, and $((E/\operatorname{Ker} k)', \mathfrak{T}_I(E/\operatorname{Ker} k)) = l^\infty$. k' is monomorphism of the space $(l^\infty, \sigma(l^\infty, l^1))$ in the space $(E', \sigma(E',E))$ with $\sigma(E',E)$-closed image $\operatorname{Ker} k^\perp$, but k' is no longer a monomorphism of the space $(l^\infty, \sigma(l^\infty,\ (l^\infty)'))$ in the space $(E', \sigma(E',E))$, since $\sigma(l^\infty, l^1) < \sigma(l^\infty, (l^\infty)')$. Therefore, although k'' is a weak homomorphism of the space $(E'', \sigma(E'', E')) = (E, \sigma(E,E'))$ in $((l^\infty)', \sigma(((l^\infty)', l^\infty)))$ by virtue of statement c) of Proposition 2.7.15, it no longer has a $\sigma((l^\infty)', l^\infty)$-closed image in $(l^\infty)'$. From Corollary 2 of Theorem 2.7.17 we also obtain that k'' is strong homomorphism of the Fréchet space $E'' = E$ in $((l^\infty)',\ \beta((l^\infty)', l^\infty)) = (l^\infty)'$ with a strongly closed and weakly dense image.

Example 2. Consider the bornological (DF)-space $(F, \mathfrak{T})$. By virtue of Theorem 5 ([82], p. 403), it is represented in the form of inductive limit of the increasing sequence of normed spaces $\{(F_n, \|\cdot\|_n)\}$ with respect to the map $I_{nm} \colon (F_n, \|\cdot\|_n) \to (F_m, \|\cdot\|_m)$ $(n \leq m)$. Then the space $(F, \mathfrak{T})$ is isomorphic to the quotient space $\bigoplus\limits_{n\in\mathbb{N}} F_n/H$ of the sums $\bigoplus\limits_{n\in\mathbb{N}} F_n$ over a closed subspace H spanned by elements of the form $x - I_{nm}(x)$, where $x \in E_n$ $(n \in \mathbb{N})$. Let $k : E = \bigoplus\limits_{n\in\mathbb{N}} F_n \to E/H = F$ be a canonical homomorphism, then k is a $\mathfrak{T}_{b^*}$-homomorphism (corollary of Theorems 2.7.3). Further, if the space $(F, \mathfrak{T})$ is barrelled, then k will be a strong homomorphism (corollary of Theorem 2.7.2). k is also a $\mathfrak{T}_k$-homomorphism (Corollary 1 of Theorem 2.7.4), since bornological space is a Mackey space by virtue of Theorem 1 ([82], p. 379). Then the operator k is also $\mathfrak{T}^\times$-homomorphism.

The adjoint mapping k' is a strong monomorphism by Corollary 1 of Theorem 2.7.2. Using the results of Section 2.7.2, one can prove that k' is also a monomorphism in some other topologies of dual spaces. In particular, k' is a $\mathfrak{T}_k$-monomorphism and a $\mathfrak{T}_I$-monomorphism. The second adjoint mapping k''

is a strong homomorphism if and only if the space $(H^{\perp},\ \beta(E',E)\cap H^{\perp}) = \big((E/H)',\beta((E/H)',(E/H))\big)$ is a distinguished Fréchet space. The example given in [6] shows that the space $(H^{\perp},\ \beta((E/H)',E/H))$ is not always distinguished. Therefore, there is an example of canonical homomorphism of the bornological (DF)-space whose second adjoint is no longer a strong homomorphism, although a strong homomorphism is its first adjoint mapping. By virtue of Proposition 14 from [65], we also obtain that if $(F,\mathfrak{T})$ is bornological (DF)-space satisfying the strict Mackey condition, then the mapping k'' is a strong homomorphism and so is its arbitrary adjoint mapping. The same result occurs if $(F,\mathfrak{T})$ is a strict (LB)-space.

Example 3. Let $(F,\mathfrak{T})$ be a non-bornological (DF)-space and $F_1 = (F,\mathfrak{T}^{\times})$, where $\mathfrak{T}^{\times}$ is the associated bornological topology ($(F,\mathfrak{T})$ can be quasi-barreled). By virtue of the results of Example 2, we have that

$$F_1 = (F,\mathfrak{T}^{\times}) = (E/\operatorname{Ker} k_1, \mathfrak{T}_1/\operatorname{Ker} k_1),$$

where $(E,\mathfrak{T}_1)$ is the sum of normed spaces and $k_1 : E \to (E/\mathrm{Ker}\, k_1, \mathfrak{T}_1/\operatorname{Ker} k_1) = F_1$ is canonical mapping. Obviously, k_1' is a strong monomorphism with the weakly closed image $k_1'(F_1') = \operatorname{Ker} k_1^{\perp}$. We also define the mapping $k_2 : E \to F$ by the equality $k_2 x = k_1 x$ for all $x \in E$. The mapping k_2 is continuous mapping of the bornological (DF)-space $(E,\mathfrak{T}_1)$ on the (DF)-space $(F,\mathfrak{T})$, which is not open. The mapping k_2 will not be a weak homomorphism if the space $(F,\mathfrak{T})$ is not a Mackey space, since then by Corollary 1 of Theorem 2.7.4, it would be a homomorphism.

Let $I : (F,\mathfrak{T}^{\times}) \to (F,\mathfrak{T})$ be the identity mapping, then $k_2 = I \circ k_1$. Further, $k_2' = k_1' \circ I'$ is a monomorphism of the space $(F',\sigma(F',F))$ into $(E',\sigma(E',E))$, but $k_2'(F')$ is not $\sigma(E',E)$-closed in E' and therefore not coincides with its $\sigma(E',E)$-closure $\operatorname{Ker} k_2^{\perp} = \operatorname{Ker} k_1^{\perp} = k_1'(F_1')$ in E'. Let us prove that if $k_2'(F') \not\subseteq k_1'(F_1')$ ([65], see also [82], p. 388), then $k_2'(F')$ is strongly closed in $k_1'(F_1')$, i.e. $(F',\beta(F',F))$ is closed in $(F_1',\beta(F_1',F_1))$ and it is a closed subspace of the space $(E',\beta(E',E))$. Let $\{B_n\}$ be the fundamental sequence of bounded sets in $(F,\mathfrak{T})$ and $B_{n,1} = I^{-1}(B_n)$, i.e. $B_n = I(B_{n,1})$. It is obvious that $\{B_{n,1}\}$ is again the fundamental sequence in $(F,\mathfrak{T}^{\times})$. Moving in the polar F', we obtain that $B_n^{\circ} = I(B_{n,1})^{\circ} = I'^{(-1)}(B_{n,1}^{\circ})$, i.e. $I'(B_n^{\circ}) = B_{n,1}^{\circ} \cap I'(F')$. This means that $\beta(F_1',F_1)$ induces the topology $\beta(F',F)$ on F' and therefore F' is closed in $(F_1',\beta(F_1',F_1))$, i.e. I' is a strong monomorphism with a strongly closed image in F_1'. Hence, the mapping I is an example of a continuous mapping of the (DF)-spaces onto the same space whose dual is a strong monomorphism.

Example 4. It is well known that the Fréchet space $(E,\mathfrak{T}_1)$ is isomorphic to the closed subspace of the product of Banach spaces $(F,\mathfrak{T}_2)$. Let $J : (E,\mathfrak{T}_1) \to$

$(F, \mathfrak{T}_2)$ be a specified monomorphism. The adjoint mapping J' is a strong homomorphism if and only if $(E, \mathfrak{T}_1)$ is distinguished. In this case, J'' is also a strong monomorphism. At the same time, J'' has a weakly closed image and, by virtue of statement d) of Theorem 2.7.15, the equalities $J(E)'' = J(E)^{\perp\perp} = (\operatorname{Ker} J')^{\perp} = J''(E'')$ are valid. By statement c) of Theorem 2.7.16, $(J(E)'',$ $\beta((F'/\operatorname{Ker} J')', J(E)') = (J''(E''), \beta(F'', F') \cap J''(E''))$. If consider as $(E, \mathfrak{T}_1)$ a distinguished Fréchet space whose second adjoint is not distinguished [24], then J''' will no longer be a strong homomorphism.

Now, let $(E, \mathfrak{T})$ be the nondistinguished Fréchet space from [65]. As is known, in this case J' is not a strong homomorphism, but J'' is a strong monomorphism with the strongly but not weakly closed image. Indeed, due to the properties of the monomorphisms J and J'', we have that $J(E)'' = \big(J(E)',\ \beta(J(E)',\ J(E))\big)' = \big(F'/\operatorname{Ker} J',\ \beta(J(E)',\ J(E))\big)' \subset (F'/\operatorname{Ker} J', \beta(F', F)/\operatorname{Ker} J')' = (\operatorname{Ker} J')^{\perp} = \overline{J''(E'')}$. Moreover, this inclusion is proper since $E' = F'/\operatorname{Ker} J'$ is bornological in the factor topology $\beta(F', F)/\operatorname{Ker} J' = \beta(F', F)^{\times}$. In the topologies $\beta(E', E)$ it is not such because, due to [65], on E' there is a linear functional that is not strongly continuous, but bounded on every bounded set, i.e. continuous in the quotient topology.

Example 5. Let now $(F, \mathfrak{T}) = s \cdot \lim\limits_{\rightarrow}(F_n, \mathfrak{T}_n)$ be strict (LF)-space. Then the space $(F, \mathfrak{T})$ is isomorphic to the quotient space $\bigoplus\limits_{n \in \mathbb{N}} F_n/H$ of the sum $E = \bigoplus\limits_{n \in \mathbb{N}} F_n$ over the closed subspace H spanned by elements of the form $x - I_{nm}x$, where $x \in F_n,\ I_{nm} : F_n \to F_m$ is the identity mapping $(n \le m)$. Let $k : E \to E/H = s \cdot \lim\limits_{\rightarrow}(F_n, \mathfrak{T}_n)$ be a canonical homomorphism. Then k is a strong homomorphism by virtue of the corollary of Theorem 2.7.2, and by virtue of the corollary of Theorem 2.7.4 k is a $\mathfrak{T}_k$-homomorphism. The adjoint mapping k' is a strong homomorphism. Indeed, every bounded set of the quotient space is contained in some F_n and, therefore, is contained in the canonical image of some bounded set from E. This is equivalent to the fulfilling the conditions of Theorem 2.7.8.

Using the similar reasoning, one can prove that k' is $\mathfrak{T}_k$ and $\mathfrak{T}_c$-homomorphism. It is more difficult to determine whether the second adjoint mapping k'' is a strong homomorphism. By virtue of statement c) Theorem 2.7.16 k'' is a strong homomorphism if and only if

$$\begin{aligned}\big((E/\operatorname{Ker} k)'', \beta((E/\operatorname{Ker} k)'', (E/\operatorname{Ker} k)')\big)& \\ = (E''/\operatorname{Ker} k'',\ \beta(E'', E')\ /\operatorname{Ker} k'),&\end{aligned}$$

i.e. $(F'', \beta(F'', F')) = (E''/\operatorname{Ker} k'', \beta(E'', E')/\operatorname{Ker} k'')$. Since $(E'', \beta(E'', E')) = \bigoplus\limits_{n \in \mathbb{N}} (F_n'', \beta(F_n'', F_n'))$, then this is equivalent to the space $(F'', \beta(F'', F'))$ being barreled or bornological, i.e. so that $(F'', \beta(F'', F')) = s \cdot \lim\limits_{\rightarrow}(F''_n, \beta(F''_n, F'_n))$ by virtue of [65]. Example of the strict (LF)-spaces constructed in [23] shows that $(F'', \beta(F'', F'))$ is not always barrelled and bornological, i.e. k'' is not always a strong homomorphism.

2.7.5 Sufficient conditions for openness and strong openness of a weakly open operator

In the work of F. Browder [29], open operators with closed graph are studied, which he again called as homomorphisms. We will apply here our results to obtain the conditions of strong openness of weakly open operator, i.e. conditions under which the implication e) ⇒ d) in Theorem 2.1 from [29] is true.

Theorem 2.7.19. *Let $T : E \to F$ be a weakly open operator of the LCS $(E, \mathfrak{T}_1)$ in the LCS $(F, \mathfrak{T}_2)$ with the dense domain $D(T)$ and closed kernel $\operatorname{Ker} T$. T is strongly open if the following conditions are fulfilled:*

a) *on the quotient space $D(T)/\operatorname{Ker} T$ the topologies $(\beta(E, E') \cap D(T))/\operatorname{Ker} T$ and $\beta(D(T)/\operatorname{Ker} T, \operatorname{Ker} T^{\perp})$ coincide;*

b) *on the image $\operatorname{Im} T$, the topologies $\beta(F, F') \cap \operatorname{Im} T$ coincide and $\beta(\operatorname{Im} T, F'/\operatorname{Im} T^{\perp})$ coincide.*

Proof. It is well known that T is weakly open if and only if in its canonical expansion the linear bijection $\check{T} : (D(T)/\operatorname{Ker} T, \sigma(E, E') \cap D(T)/\operatorname{Ker} T) \to (\operatorname{Im} T, \sigma(F, F') \cap \operatorname{Im} T)$ is weakly open. Due to the properties of the weak topology on the quotient space $D(T)/\operatorname{Ker} T$, the topologies $\sigma(F, F') \cap D(T)/\operatorname{Ker} T$ and $\sigma(D(T)/\operatorname{Ker} T, \operatorname{Ker} T^{\perp})$ coincide. Also, on $\operatorname{Im} T$ the topologies $\sigma(F, F') \cap \operatorname{Im} T$ and $\sigma(\operatorname{Im} T, F'/\operatorname{Im} T^{\perp})$ coincide. Therefore, $\check{T}$ is weakly open as a mapping

$$\check{T} : (D(T)/\operatorname{Ker} T,\ \sigma(D(T)/\operatorname{Ker} T,\ \operatorname{Ker} T^{\perp})) \to (\operatorname{Im} T,\ \sigma(F, F') \cap \operatorname{Im} T),$$

i.e. $\check{T}^{-1}$ is weakly and therefore strongly continuous as a mapping

$$\check{T}^{-1} : \beta(\operatorname{Im} T,\ F'/\operatorname{Im} T^{\perp}) \to (D(T)/\operatorname{Ker} T,\ \beta(D(T)/\operatorname{Ker} T,\ \operatorname{Ker} T^{\perp})).$$

Taking into account conditions a) and c) we obtain the validity of the following diagram:

$$\begin{tikzcd}
(D(T)/\operatorname{Ker} T,\ \beta(E, E') \cap D(T)/\operatorname{Ker} T) \arrow[d, shift right] & (\operatorname{Im} T,\ \beta(F, F') \cap \operatorname{Im} T) \arrow[d, shift right] \\
(D(T)/\operatorname{Ker} T,\ \beta(D(T)/\operatorname{Ker} T,\ \operatorname{Ker} T^{\perp})) \arrow[u, shift right] & (\operatorname{Im} T,\ \beta(\operatorname{Im} T, F'/\operatorname{Im} T^{\perp})), \arrow[l] \arrow[u, shift right]
\end{tikzcd}$$

where the arrows indicate the continuous mappings. From here we get that T is strongly open.

It should be noted that to prove the converse statement to Theorem 2.7.19, i.e. to prove the coincidence of the above topologies on $D(T)/\operatorname{Ker} T$ and $\operatorname{Im} T$ for strongly and weakly open operator T, in the a priori conditions of Theorem 2.7.19 one must additionally require that T is a weak homomorphism. Indeed, in this case, due to the well-known inequalities between strong topologies on $D(T)/\operatorname{Ker} T$ and $\operatorname{Im} T$, we obtain the validity of the following diagram:

$$\begin{tikzcd}
(D(T)/\operatorname{Ker} T,\ \beta(E,E')\cap D(T)/\operatorname{Ker} T) \arrow[d] & (\operatorname{Im} T,\ \beta(F,F')\cap \operatorname{Im} T) \arrow[l, "\check{T}"'] \\
(D(T)/\operatorname{Ker} T,\ \beta(D(T)/\operatorname{Ker} T,\ \operatorname{Ker} T^{\perp})) \arrow[r, shift left, "\check{T}"] & (\operatorname{Im} T,\ \beta(\operatorname{Im} T, F'/\operatorname{Im} T^{\perp})), \arrow[l, shift left] \arrow[u]
\end{tikzcd}$$

where the operators indicated by arrows are continuous. It follows that two identical operators in the diagram are isomorphisms, i.e. the above strong topologies on $D(T)/\operatorname{Ker} T$ and $\operatorname{Im} T$ coincide. $\square$

Let us now apply Theorem 2.7.2 to prove the sufficient conditions for a weakly open operator to be open and strongly open.

Theorem 2.7.20. *Let $T : E \to F$ be a weakly open linear operator of LCS $(E, \mathfrak{T}_1)$ in LCS $(F, \mathfrak{T}_2)$. If the space $(\operatorname{Im} T,\ \mathfrak{T}_2 \cap \operatorname{Im} T)$ is Mackey space, then T is open. If, moreover, the graph of the operator $G(T)$ is barrelled, then the operator T is strongly open.*

Proof. For the linear operator T, consider the continuous operator $S : G(T) \to F$, defined by the equality $S(e, Te) = Te$, $e \in D(T)$. According to ([29], Theorem 2.2), T is weakly (resp. strongly) open if and only if S is weakly (resp. strongly) open. Therefore, to prove our statement it is sufficient to prove that S is a strong homomorphism. Really, we have the following expansion of the operator $S = J\check{S}K$:

$$\begin{tikzcd}
(G(T)/\operatorname{Ker} S, (\mathfrak{T}_1\times\mathfrak{T}_2)/\operatorname{Ker} S) \arrow[r, "\check{S}"] & (\operatorname{Im} T, \mathfrak{T}_2\cap\operatorname{Im} T) \arrow[r, shift left, "id_1"] & (\operatorname{Im} T, \tau(\operatorname{Im} T, F'/\operatorname{Im} T^{\perp})) \arrow[l, shift left] \arrow[d, shift left, "\check{S}"] \\
(G(T)/\operatorname{Ker} S, \tau(G(T), G(T)')/\operatorname{Ker} S) \arrow[u, "id_2"] \arrow[rr, shift left, "id_3"] & & (G(T)/\operatorname{Ker} S, \tau(G(T)/\operatorname{Ker} S, \operatorname{Ker} S^{\perp})) \arrow[ll, shift left] \arrow[u, shift left]
\end{tikzcd}$$

where each operator indicated by arrows is continuous. Indeed, id_1 is an isomorphism due to the fact that $\mathfrak{T}_2 \cap \operatorname{Im} T$ is a Mackey topology. id_2 is continuous due to the fact that the original topology is weaker than the Mackey topology. id_3 is an isomorphism due to the fact that the quotient topology $\tau(G(T), G(T)')/\operatorname{Ker} S$ of the Mackey topology coincides with the Mackey topology $\tau(G(T)/\operatorname{Ker} S, \operatorname{Ker} S^{\perp})$ on $G(T)/\operatorname{Ker} S$. $\check{S}_1$ is the isomorphism due to the fact that weak isomorphisms are also isomorphisms in Mackey topologies. From this it already turns out that S is a homomorphism of the space $(G(T), (\mathfrak{T}_1 \times \mathfrak{T}_2) \cap G(T))$ in $(F, \mathfrak{T}_2)$, i.e. T is open.

Further, barreledness of the space $(G(T), (\mathfrak{T}_1 \times \mathfrak{T}_2) \cap G(T))$ means the coincidence of the topologies $(\mathfrak{T}_1 \times \mathfrak{T}_2) \cap G(T)$ and $\beta(G(T), G(T)')$ on $G(T)$. Next, from the fact that S is a homomorphism, it follows that the spaces $(G(T)/\operatorname{Ker} S$, $(\mathfrak{T}_1 \times \mathfrak{T}_2)/\operatorname{Ker} S)$ and $(\operatorname{Im} T,\ \mathfrak{T}_2 \cap \operatorname{Im} T)$ are barrelled and therefore $(\mathfrak{T}_1 \times \mathfrak{T}_2)/\operatorname{Ker} S) = \beta(G(T)/\operatorname{Ker} S, \operatorname{Ker} S^{\perp})$ and $\mathfrak{T}_2 \cap \operatorname{Im} T = \beta(\operatorname{Im} T, F' / \operatorname{Im} T^{\perp})$. This means that T is strong open. □

This theorem shows that the conditions for validity of the implication e) $\Longrightarrow$ d) in ([29], Theorem 2.1) can be weakened and make it less burdensome.

Let $A : E \to F$ be a linear operator, where E and F are Fréchet spaces, the quotient space $F/\overline{\operatorname{Im} A}$ is called the *co Kernel* of the operator A and is denoted by $\operatorname{coker} A$. In the future, dimensions will play an important role

$$\alpha(A) = \dim \ker A \quad \text{and} \quad \beta(A) = \dim \operatorname{coker} A,$$

the last of which is called the *defective number* of the operator A. The defective number $\beta(A)$ is finite if and only if the space $\operatorname{Im} A^{\perp} \subset F'$ (called the *defective space* of the operator A) is finite-dimensional, in which case both dimensions are the same.

In the case of a dense domain of definition of $D(A)$, the following statement holds.

The numbers $\beta(A)$ and $\alpha(A^*)$ are both finite or infinite at the same time, and if they are finite, then

$$\beta(A) = \alpha(A^*). \tag{2.7.1}$$

An ordered pair of numbers $(\alpha(A), \beta(A))$ is called the *d-characteristic* of the operator A. If at least one of the numbers $\alpha(A)$ and $\beta(A)$ is finite, then the difference $\operatorname{Ind} A = \alpha(A) - \beta(A)$ is called the *index* of the operator A.

An operator A is said to have *finite d-characteristic* or *finite index* if both $\alpha(A)$ and $\beta(A)$ are finite.

A closed normally solvable operator A is called a Noetherian or Φ-operator if its d-characteristic is finite.

The set of all Φ-operators $A : E \to F$ is denoted by $\Phi(E, F)$.

Corollary. *Let E and F be Fréchet spaces and $T : E \to F$ be a weakly open linear Φ-operator. Then T is open and strongly open.*

Proof. The space $(\operatorname{Im} T, \mathfrak{T}_2 \cap \operatorname{Im} T)$ is a Mackey space since $(\operatorname{Im} T, \mathfrak{T}_2 \cap \operatorname{Im} T)$ is a Fréchet space. Therefore, T is open. Since the graph is closed, we also obtain that T is strongly open. $\square$

CHAPTER 3

Linear problems with a sequence of problem elements sets

This chapter deals with the linear problems with a sequence of problem elements sets. In this regard, a generalization of Minkowski functional is given. Next, in terms of strong proximality in the Hilbert spaces, the definition of an interpolation spline with non-adaptive information with respect to the known metrics is given. Questions about the existence of an interpolation spline in Fréchet spaces, that is, strong best approximations in subspaces of finite codimension, are studied. The well known theorems of James and Bishop–Phelps are generalized. It is proved that an interpolation spline for arbitrary information of cardinality 1 exists if and only if the space is reflexive quojection (strictly regular). The classes of reflexive Fréchet spaces for which interpolation splines do not exist in subspaces of codimension 1 are indicated. The classes of reflexive Fréchet spaces in which interpolation splines exist for information of any cardinality are also indicated.

3.1 Definition of a spline and spline algorithm in Fréchet spaces

Let F_1 be a linear space, G be a metrizable LCS, $S : F_1 \to G$ be a linear solution operator, and $\{V_n\}$ be a non-increasing sequence of absolutely convex subsets of the space F_1, i.e. $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq \cdots$. Consider the sets

$$K_r = rV_n, \quad \text{where} \quad r \in I_n = \begin{cases} [1, \infty[& \text{for} \quad n = 1, \\ [2^{-n+1}, 2^{-n+2}[& \text{for} \quad n \geq 2. \end{cases}$$

Let $\mu_{\{V_n\}}$ be the functional defined on F_1 as

$$\mu_{\{V_n\}}(f) = \inf\{r > 0;\ f \in K_r\}. \tag{3.1.1}$$

If $V_1 = V_2 = \cdots = V_n = \cdots = F$, then $K_r = rF$ and $\mu_{\{V_n\}}$ coincides with the Minkowski functional μ_F of F. Further, we will call the functional $\mu_{\{V_n\}}$ as the

Minkowski functional for the sequence $\{V_n\}$. We give the following properties of this functional (3.1.1):

1. It is clear that $\mu_{\{V_n\}}(f) \geq 0,\ f \in F_1$.

2. If $f_1, f_2 \in F_1$, then $\mu_{\{V_n\}}(f_1 + f_2) \leq \mu_{\{V_n\}}(f_1) + \mu_{\{V_n\}}(f_2)$. To prove this fact, we first note that $K_r + K_s \subset K_{r+s}$. Consider the following three special cases: a) $r \in [2^{-n}, 2^{-n+1}[$ and $s \in [2^{-m}, 2^{-m+1}[$, where $1 \leq n \leq m$. Then $r+s \in [2^{-n}+2^{-m}, 2^{-n+1}+2^{-m+1}[\subset [2^{-n}, 2^{-n+1}[\cup[2^{-n+1}, 2^{-n+2}[$. If $r+s \in [2^{-n}, 2^{-n+1}[$, then $K_r+K_s = rV_{n+1}+sV_{m+1} \subset rV_{n+1}+sV_{n+1} = (r+s)V_{n+1} = K_{r+s}$. If $r+s \in [2^{-n+1}, 2^{-n+2}[$, then $K_r+K_s = rV_{n+1}+sV_{m+1} \subset (r+s)V_n = K_{r+s}$; b) Let $r \in [1, \infty[$ and $s \in [2^{-m}, 2^{-m+1}[,\ m \in \mathbb{N}$, then $r+s \in [1, \infty[$, and we have $K_r + K_s = rV_1 + sV_{m+1} \subset (r+s)V_1 = K_{r+s}$; c) If $r, s \in [1, \infty[$, then it is clear that $K_r + K_s \subset K_{r+s}$. Now, let $\mu_{\{V_n\}}(f_1) = r$ and $\mu_{\{V_n\}}(f_2) = s$. Then for a sufficiently small $\varepsilon > 0$ we will have $f_1 \in K_{r+\varepsilon/2}$ and $f_2 \in K_{s+\varepsilon/2}$, i.e. $\mu_{\{V_n\}}(f_1 + f_2) \leq r + s + \varepsilon$ for sufficiently small $\varepsilon > 0$ and, consequently, $\mu_{\{V_n\}}(f_1 + f_2) \leq r + s = \mu_{\{V_n\}}(f_1) + \mu_{\{V_n\}}(f_2)$.

3. If $\cap_{r\in\mathbb{R}^+}K_r \neq \{0\}$, then $\operatorname{Ker}\mu_{\{V_n\}}(\cdot) \neq \{0\}$ and $\mu_{\{V_n\}}(f-g) = d(f,g)$ is a translation invariant submetric on F_1. If $\cap_{r\in\mathbb{R}^+}K_r = \{0\}$, then $Ker\mu_{\{V_n\}}(\cdot) = \{0\}$ and $\mu_{\{V_n\}}(f-g) = d(f,g)$ is a metric on F_1. Really, it is easy to see that if $x_0 \in \cap_{r\in\mathbb{R}^+}K_r$ and $x_0 \neq 0$, then $\mu_{\{V_n\}}(x_0) = 0$.

We denote the Minkowski functional of K_r by $q_r(\cdot)$. It is clear that if $r \in I_n$, then $q_r(\cdot) = r^{-1}\|\cdot\|_n$, where $\|\cdot\|_n$ is the Minkowski functional for V_n.

Let $F_1 = \operatorname{Ker}\mu_{\{V_n\}} + \operatorname{Ker}\mu^{\perp}_{\{V_n\}}$, where the second summand is the algebraic complement linear subspace of $\operatorname{Ker}\mu_{\{V_n\}}$ in F_1. For any $f, g \in F_1$, $f = f_1 + f_2$ and $g = g_1 + g_2$, where $f_1, g_1 \in \operatorname{Ker}\mu_{\{V_n\}}$ and $f_2, g_2 \in \operatorname{Ker}\mu^{\perp}_{\{V_n\}}$. Define $E = \operatorname{Ker}\mu^{\perp}_{\{V_n\}}$ and for $f_2, g_2 \in E$, $d(f_2, g_2) = \mu_{\{V_n\}}(f-g)$. If $\mu_{\{V_n\}}(f_2) = 0$, then $f_2 \in \operatorname{Ker}\mu_{\{V_n\}}$ and $f_2 = 0$. The functional $d(f_2, g_2) = \mu_{\{V_n\}}(f-g)$ is a metric on E and E is a linear metrizable LCS. Define a linear operator T as $Tf = f_2$. In fact, T is an algebraic projection of the space F_1 onto the subspace $\operatorname{Ker}\mu^{\perp}_{\{V_n\}}$. It follows that

$$d(Tf, Tg) = d(f_2, g_2) = \mu_{\{V_n\}}(f-g). \tag{3.1.2}$$

The Minkowski functional of K_r is denoted by $q_r(\cdot)$. It is clear that if $r \in I_n$, then $q_r(\cdot) = r^{-1}\|\cdot\|_n$, where $\|\cdot\|_n$ is the Minkowski functional for V_n. If $r > 0$, $\overline{q_r}$ is the Minkowski functional of the set $\{Tx \in E;\ d(Tx, 0) \leq r\}$.

Let $I : F_1 \to \mathbb{R}^m$ be a non-adaptive information of cardinality m and $y \in I(F_1)$. $T : F_1 \to E$ is the above-mentioned linear operator. An element $\sigma = \sigma(y)$ is called a spline interpolatory y (briefly, a spline) if and only if

(i) $I(\sigma) = y$,

(ii) $d(T\sigma, 0) = \inf\{d(Tz, 0);\ z \in F_1 \text{ and } I(z) = y\} = r$,

(iii) $\overline{q_r}(T\sigma) = \inf\{\overline{q_r}(Tz);\ z \in F_1$ and $I(z) = y\}$ if $r > 0$, where $\overline{q_r}$ is the Minkowski functional of the set $\{Tx \in E;\ d(Tx, 0) \leq r\}$.

The above definition deals with the problem of existence of minimum of functionals on the set $\{z \in F_1;\ I(z) = y\}$ the closedness of which can be asserted, since the space F_1 is, in general, only linear. This is also the case in the classical definition [11]. Let us prove that

$$\overline{q_r}(Tx) = q_r(x),\ x \in F_1,\ r > 0. \tag{3.1.3}$$

By the definition of the functional $\mu_{\{V_n\}}$, we find that if $\mu_{\{V_n\}}(x) \leq r$, then $x \in K_{r+\varepsilon}$ for all $\varepsilon > 0$, i.e. $x \in (r+\varepsilon)V_n$, when $r \in I_n$. This implies that $\|x\|_n \leq r+\varepsilon$ for arbitrary $\varepsilon > 0$, i.e., $\|x\|_n \leq r$. Analogously, we conclude that if $\|x\|_n \leq r$, then $\mu_{\{V_n\}} \leq r$. That is, $\mu_{\{V_n\}} \leq r \Leftrightarrow \|x\|_n \leq r$. Further, we have that $\overline{q_r}(Tx) = \inf\{\alpha > 0,\ d(Tx/\alpha, 0) \leq r\} = \inf\{\alpha > 0,\ \mu_{\{V_n\}}(Tx/\alpha) \leq r\}$. Let α_0 be an arbitrary number with the property $\alpha_0 > \overline{q_r}(Tx)$. Then $d(Tx/\alpha_0) \leq r$, i.e. $\mu_{\{V_n\}}(Tx/\alpha_0) \leq r$ and $\|Tx/\alpha_0\| \leq r$. This means that $\|x/\alpha_0\|_n \leq r$, i.e., $q_r(x) \leq \alpha_0$. That is, $q_r(x) \leq \overline{q_r}(Tx)$. On the other hand, if β is an arbitrary number such that $x/\beta \in K_r$, then $\|x/\beta\|_n \leq r$. So, $\|Tx/\beta\|_n \leq r$. This means that $\mu_{\{V_n\}}(Tx/\beta) \leq r$, that is, $d(Tx/\beta, 0) \leq r$. Therefore, $\overline{q_r}(Tx) \leq q_r(x)$. Thus, (3.1.3) is proved.

Let F_1 be a linear space and μ be a nonnegative functional for which the sets $\{x \in F_1;\ \mu(x) \leq r\}$, $r \in \mathbb{R}^+$, are absolutely convex. Denote by q_r the Minkowski functional of this set. We say that a subspace $M \subset F_1$ is strongly proximal in F_1 with respect to μ if for arbitrary $x \in F_1$, there exists $h^* \in M$ such that $\inf\{\mu(x - h),\ h \in M\} = \mu(x - h^*) = r$, and if $r > 0$, then $\inf\{q_r(x - h),\ h \in M\} = q_r(x-h^*)$. We call such $h^* \in M$ the strong best approximation element for $x \in F_1$ in M. We call a subset M of the space F_1 strongly proximal if for each $x \in F_1$ in M, there is an element of the strong best approximation. The definition of strong proximality was introduced in [4].

Theorem 3.1.1. *Let $y \in I(F_1)$, $T : F_1 \to E$ be the above mentioned linear operator and I be a non-adaptive information of a cardinality $m \in \mathbb{N}$. Then there exists a generalized spline interpolatory y if and only if the subspace* $\operatorname{Ker} I$ *is strongly proximal in F_1 with respect to the functional $\mu_{\{V_n\}}$.*

Proof. First, we assume that $\operatorname{Ker} I$ is strongly proximal in F_1 with respect to the functional $\mu_{\{V_n\}}$. Let f be an arbitrary element belonging to the set $I^{-1}(y)$. Then we have

$$\inf\{\mu_{\{V_n\}}(f - h) :\ h \in \operatorname{Ker} I\} = \mu_{\{V_n\}}(f - h^*) = r$$

and, if $r > 0$, then

$$\inf\{q_r(f - h) :\ h \in \operatorname{Ker} I\} = q_r(f - h^*)$$

for some $h^* \in \operatorname{Ker} I$. Denote $\sigma = f - h^*$. By the property (3.1.2) of the metric d, we have

$$\begin{aligned}\inf\{\mu_{\{V_n\}}(f-h):\ h \in \operatorname{Ker} I\} &= \mu_{\{V_n\}}(f-h^*) = r = \mu_{\{V_n\}}(\sigma)\\ &= d(T\sigma, 0) = \inf\{d(Tz,0);\ z \in F_1,\ I(z) = y\}.\end{aligned}$$

From the above and (3.1.3), we have

$$\begin{aligned}\inf\{q_r(f-h):\ h \in \operatorname{Ker} I\} &= q_r(f-h^*) = \overline{q_r}(T\sigma)\\ &= \inf \overline{q_r}(Tz);\ z \in F_1,\ I(z) = y\}.\end{aligned}$$

Conversely, let f be an element in F_1, $If = y$ and σ be a generalized spline interpolatory y. We represent an element $z \in I^{-1}(y)$ in the form $z = f - h$, where $h \in \operatorname{Ker} I$, and consider the element $h^* = f - \sigma \in \operatorname{Ker} I$. It is clear that $\sigma = f - h^*$ satisfies (i)–(iii). Therefore, h^* is a strongly best approximation for f in $\operatorname{Ker} I$. □

In the sequel, we will assume that F_1 is an LCS with a non-increasing sequence of absolutely convex closed zero-neighborhoods $\{V_n\}$. In particular, such a sequence of absolutely convex closed neighborhoods exists if F_1 is a metrizable LCS. In this case, T is an identity operator and the space (E, d) is the linear metric LCS in which linear operations are continuous. The existence of such a metric in the strict (LF)-space is proved in Section 2.6. The generalization of this result for strict inductive limits of LCS, on which there exists metrics, is proved by S. Dierolf and K. Floret [10]. In that case, T is a continuous imbedding from F_1 into E.

If, moreover, $\{V_n\}$ is a local basis of non-increasing sequence of neighborhoods of zero for some topology, then $\mu_{\{V_n\}}(f-g) = d(f,g)$ is the continuous metric generating the topology defined by the sequence $\{V_n\}$. This functional is quasiconvex, i.e., the sets $\{x:\ \mu_{\{V_n\}}(x) \le r\}$, $r \in R^+$, are absolutely convex and coincide with K_r. Topological boundary $\partial K_r = \{x \in F_1;\ q_r(x) = 1\}$ of K_r coincides with the metric boundary $\{x \in F_1;\ \mu_{\{V_n\}}(x) = r\}$ for $r \in \operatorname{int} I_n$ and they, in general, differ for $r = 2^{-n+1}$ $(n \in \mathbb{N})$ (see Section 2.5).

Below we will often replace an arbitrary translation invariant metric d by quasinorm $|\cdot|$ (i.e., we will replace $d(x,y)$ by $|x-y|$).

Let $\|\cdot\|_n$ be the Minkowski functional of V_n. The definition of the functional $\mu_{\{V_n\}}$ for the sequence $\{V_n\}$ coincides with the quasinorm of the metric (2.5.12) and has the following form:

$$|x| = \begin{cases} \|x\|_1, & \text{when } \|x\|_1 \geq 1 \ , \\ 2^{-n+1}, & \text{when } \|x\|_n \leq 2^{-n+1} \text{ and} \\ & \qquad \|x\|_{n+1} \geq 2^{-n+1} \ (n \in \mathbb{N}), \\ \|x\|_{n+1}, & \text{when } 2^{-n} \leq \|x-y\|_{n+1} < 2^{-n+1} \ \ (n \in \mathbb{N}), \\ \ \ 0, & \text{when } x = 0 \ . \end{cases} \tag{3.1.4}$$

It should be noted that in (3.1.4), under $\|\cdot\|_n$ we mean the Minkowski functional for V_n, and we will keep this in mind throughout what follows. Since $q_r(\cdot) = r^{-1}\|\cdot\|_n$ for $r \in I_n$, we find that for the metric (3.1.4), in terms of the above notation, $\sigma = f - h^*$ is a spline interpolatory y if and only if $I(\sigma) = y$,

$$d(f, \operatorname{Ker} I) = d(f, h^*) = r = d(\sigma, 0) = |\sigma|, \ \text{ when } r \in \operatorname{int} I_n \ (n \in \mathbb{N}) \tag{3.1.5}$$

and

$$\begin{aligned} E(f, \operatorname{Ker} I, V_n) &:= \inf\{\|f-h\|_n;\ h \in \operatorname{Ker} I\} \\ = \|f-h^*\|_n &= \|\sigma_n\| \leq r, \ \text{ when } \ r = 2^{-n+1} \ \ (n \in \mathbb{N}). \end{aligned} \tag{3.1.6}$$

For $V_1 = V_2 = \cdots = F$, we have proven that $K_r = rF$, $|\cdot| = \mu_F(\cdot)$, and the interpolation spline coincides with the classical one.

The number r, mentioned in the definition of a spline interpolatory (3.1.5), does not depend on the choice of $f \in F_1,\ I(f) = y, d(f, \operatorname{Ker} I) = r$. Indeed, if

$$I(f_1) = I(f_2) = y, \ \ f_2 - f_1 = z \in \operatorname{Ker} I, \ \ d(f_1, \operatorname{Ker} I) = d(f_1, h_1^*)$$

and

$$\inf\{q_r(f_1 - h);\ h \in \operatorname{Ker} I\} = q_r(f_1 - h^*),$$

we have

$$d(f_2, \operatorname{Ker} I) = d(f_2,\ h_1^* + z) = d(f_1, h_1^*)$$

and

$$\inf\{q_r(f_2 - h);\ h \in \operatorname{Ker} I\} = q_r(f_2 - h_1^* + z) = q_r(f_1 - h_1^*).$$

Proposition 3.1.2. *Let E be a metrizable LCS with a non-decreasing sequence of seminorms $\{\|\cdot\|_n\}$, with metrics* (3.1.4)*, and M be a convex subset of E. Then the following statements hold:*

a) *If $r \in \operatorname{int} I_n$ $(n \in \mathbb{N})$, then the equalities $d(x, M) = r = |x - h^*|$ and $\inf\{\|x-h\|_n;\ h \in M\} = \|x - h^*\|_n = r$, where h^* is some element of M, are equivalent.*

b) *If $d(x, M) = r = 2^{-n+1}$ $(n \in \mathbb{N})$ and $\inf\{\|x-h\|_n;\ h \in M\} = \|x - h^*\|_n = \lambda$ for some $h^* \in M$, then $d(x, M) = d(x, h^*)$.*

Proof. a) Let $d(x, M) = |x - h^*| = r \in \mathrm{int} I_n$ $(n \in \mathbb{N})$ for some $h^* \in M$. From the definition of the metric (3.1.4) it follows that $d(x, M) = |x - h^*| = \|x - h^*\|_n = r$. Let us prove that $\|x - h^*\|_n = \inf\{\|x - h\,|_n;\ h \in M\}$. Assume the opposite that $s = \|x - h_1\|_n < \|x - h^*\|_n = r$ for some $h_1 \in M$. Then, according to the properties of the metric (3.1.4), we have $d(x, h_1) = \|x - h_1\|_n < r$ if $s \in I_n$ and $d(x, h_1) \leq 2^{-n+1} < r$ if $s < 2^{-n+1}$. Thus, $d(x, h_1) < r$, which is impossible. Analogously, we can show that the inequality $\inf\{\|x - h\|_n;\ h \in M\} = \|x - h^*\|_n = r \in \mathrm{int} I_n$, $h^* \in M$, implies that $d(x, M) = |x - h^*| = r$. Thus, part a) is proved.

To prove b), let $d(x, M) = r = 2^{-n+1}$ and $\inf\{\|x - h\|_n;\ h \in M\} = \|x - h^*\|_n = \lambda \leq 2^{-n+1}$, where $h^* \in M$. Let us show that $\|x - h^*\|_{n+1} = s \geq 2^{-n+1}$. If we assume that $s < 2^{-n+1}$, then $d(x, h^*) \leq \max(s, 2^{-n}) < 2^{-n+1} = r$, which is impossible. Thus, $\|x - h^*\|_n \leq 2^{-n+1}$ and $\|x - h^*\|_{n+1} \geq 2^{-n+1}$. According to (3.1.4), this means that $d(x, h^*) = 2^{-n+1} = r$. □

It should be noted that for the metric (3.1.4), the element of the best approximation with respect to the metric may not have a similar property with respect to $q_r(\cdot)$ for $r \in 2^{-n+1}$ $(n \in \mathbb{N})$ and therefore, with respect to $\|\cdot\|_n$ (see the following examples).

Example 3.1.1. Now we give an example showing that if $d(f, G) = d(f, g_0) = 1$, then g_0 may not be an element of the best approximation with respect to the seminorm $\|\cdot\|_1$. Let $E = C(\mathbb{R})$ be the Fréchet space of continuous real-valued functions with the topology of compact convergence on $\mathbb{R}$, which is given by the sequence of seminorms $\|f\|_n = \max\{|f(t)|; t \in [-n, n]\}$, $n \in \mathbb{N}$. Let $f(t) = t^2$ and let $G = \mathcal{P}_2$ be the subspace of polynomials of degree at most 1. According to the classical Chebyshev theorem,

$$\begin{aligned}\inf\{\|f - m\|_1,\ m \in \mathcal{P}_2\} &= \inf\{\max\{|t^2 - a_1 t - a_2|,\ t \in [-1, 1]\}, a_1, a_2 \in \mathbb{R}\}\\ &= \max\{|T_2(t)|/2;\ t \in [-1, 1]\} = \max\{|t^2 - 1/2|; t \in [-1, 1]\}\\ &= \|t^2 - m_0\|_1 = 1/2,\end{aligned}$$

where $T_2(t) = 2t^2 - 1$ is the Chebysheff polynomial of the first kind and $m_0(t) \equiv 1/2$ is the unique best approximation of f with respect to the seminorm $\|\cdot\|_1$. So, $\|t^2 - m\|_1 \geq 1/2$ for all $m \in \mathcal{P}_2$. In addition, we have

$$\begin{gathered}\inf\{\|f - m\|_2,\ m \in \mathcal{P}_2\}\\ = \inf\{\max\{|t^2 - a_1 t - a_2|,\ t \in [-2, 2]\}, a_1, a_2 \in \mathbb{R}\} \geq \|t^2 - 2\|_2 = 2.\end{gathered}$$

This means that $\|t^2 - m\|_2 > 1$ for all $m \in \mathcal{P}_2$. Consider the element $m \in \mathcal{P}_2$ for which $1/2 < \|t^2 - m\|_1 \leq 1$. From the definition of the metric (3.1.4), it follows that the inequalities $\|t^2-m\|_2 \geq 1$ and $\|t^2-m\|_1 \leq 1$ are valid for the subspace $\mathcal{P}_2$ and, therefore, $d(t^2, m) = 2^{-1+1} = 1$. This means that if $1/2 \leq \|t^2 - m\|_1 \leq 1$, then $m \in \mathcal{P}_2$ is an element of the best approximation with respect to the metric, but no element will be the element of the best approximation with respect to the seminorm $\|\cdot\|_1$. Only $m_0(t) = 1/2 \in \mathcal{P}_2$ is the element of the best approximation with respect to d and $\|\cdot\|_1$ simultaneously.

Example 3.1.2. Let us now give an example of a subspace $G \subset C(\mathbb{R})$ and $x \in C(\mathbb{R})\backslash G$ such that $d(x, G) = d(x, g_0) = 2^{-1}$ and

$$\inf\{\|x - g\|_2;\ g \in G\} = \|x - g_0\|_2 = 0.$$

Indeed, let $G = \mathcal{P}_3$ be a subspace of algebraic polynomials, whose degree does not exceed 2. Let us define the function $f(t)$ as follows:

$$f(t) = \begin{cases} t^2, & \text{when } |t| \leq 2, \\ 0, & \text{when } |t| \geq 3, \\ \text{linear in the intervals } [-3, -2] \text{ and } [2, 3]. \end{cases}$$

Let us prove that $d(f, \mathcal{P}_3) = \inf\{d(f, m);\ m \in \mathcal{P}_3\} = 2^{-1}$. Indeed, if $g_0(t) = t^2 \in \mathcal{P}_3$, then $\|f - g_0\|_2 = 0$, that is, $f - g_0 \in \operatorname{int} V_2$. Further, it is easy to prove that $\|f - g_0\|_3 = 9$, that is, $f - g_0 \in \operatorname{int} V_2 \backslash 2\operatorname{int} V_3$. By the definition of the metric d, we have that $d(f, g_0) = 2$ and, therefore, $\inf\{d(f, g);\ g \in \mathcal{P}_3\} \leq 2^{-1}$. Let us assume that $d(f, G) = r \in]2^{-2}, 2^{-1}[$. Then, by virtue of Proposition 3.1.2, we have that $\inf\{q_r(f - g);\ g \in \mathcal{P}_3\} = 1$, where $q_r = r^{-1}\|\cdot\|_3$ is the Minkowski functional for K_r. From here we get

$$\inf\{\|f - g\|_3;\ g \in \mathcal{P}_3\} = r \in [2^{-2}, 2^{-3}[.$$

But, on the other hand, by virtue of the well-known Chebyshev theorem (Chebyshev alternance) we have

$$\inf\{\|f - g\|_3\ g \in \mathcal{P}_3\} = \|f - m_0\|_3 = 2,$$

where $m_0(t) = 2$. In particular, this follows from the following equalities:

$$f(-3) - 2 = -(f(-2) - 2) = f(0) - 2 = -(f(2) - 2) = f(3) - 2.$$

This means that $8 \neq 9 \cdot 4$ and, therefore, $d(f, \mathcal{P}_3) = 2^{-1}$, but $\inf\{q_{2^{-1}}(f - g);\ g \in \mathcal{P}_3\} = \inf\{\|f - g\|_2;\ g \in \mathcal{P}_3\} = 0$.

By virtue of Theorem 3.1.1, the definition of a spline for the metric (3.1.4) takes the following form: let I be non-adaptive information, $y \in I(F_1),\ I(f) = y$ and

$d(f, \operatorname{Ker} I) = r$ for some $f \in F_1$. Then $\sigma = f - h^*$ will be called a generalized interpolation spline if

$$d(f, \operatorname{Ker} I) = d(f, h^*) = r, \text{ when } r \in \operatorname{int} I_n \ (n \in \mathbb{N})$$

and

$$E(f, \operatorname{Ker} I, V_n) = \mu_{V_n}(f - h^*), \text{ when } r = 2^{-n+1} \ (n \in \mathbb{N}),$$

where $V_n = \{f \in F_1;\ \|f\|_n \leq 1\}$ is a non-increasing sequence of neighborhoods of zero, participating in the definition of the metric (3.1.4). If $V_1 = V_2 = \cdots = F$, then $K_r = rF$, $|f| = \mu_F(f)$ and the interpolation spline for the metric coincides with the classical interpolation spline ([158], p. 95).

The above definition of a spline is equivalent to the following: let us assume that $y \in I(F_1),\ I(f) = y$ for some $f \in F_1$ and $d(f, \operatorname{Ker} I) = r \in I_n$. $\sigma = f - h^*$ will be called a interpolation spline if

$$E(f, \operatorname{Ker} I, V_n) = \mu_{V_n}(f - h^*), \quad d(f, \operatorname{Ker} I) = r \in I_n.$$

So, to find the interpolation spline $\sigma(y)$, we find the classical interpolation spline for the set $F = V_n$ in the case $r \in \operatorname{int} I_n$. This means that the interpolation spline σ_n depends on n for different y. More precisely, $\sigma(y) = \sigma_n(y)$ when $I(f) = y$ and $d(f, \operatorname{Ker} I) = r \in \operatorname{int} I_n$.

The interpolation spline $\sigma(y)$ does not depend on n in the case where the finite codimension subspace $\operatorname{Ker} I$ has an orthogonal complement in F_1 with respect to the topology generated by the sequence of hilbertian seminorms $\{\| \cdot \|_n\}$. This follows from the fact that in this case the best approximation in $\operatorname{Ker} I$ does not depend on n and is the same for all seminorms $\| \cdot \|_n$. In what follows, we will mainly consider just such cases.

For metrics (2.5.2) and (2.5.4), the equalities $d(f, G) = d(f, h^*) = r$ and $\inf\{q_r(f - h) : h \in G\} = q_r(f - h^*) = 1$ are equivalent for an arbitrary closed subset G of the space F_1, that is, for these metrics the concepts of proximality and strong proximality coincide.

Lemma 3.1.3. *Let* (E, d_2) *be a metrizable LCS with the metric* (2.5.4) *and* $G \subset E$. *Then for* $f \notin \overline{G}$, *the equalities*

$$\inf\{d_2(f - g) : g \in G\} = r > 0 \tag{3.1.7}$$

and

$$\inf\{q_r(f - g) : g \in G\} = 1 \tag{3.1.8}$$

are equivalent.

Proof. Let us assume that the equality (3.1.3) is satisfied, but

$$\inf\{q_r(f-g);\ g \in G\} = \lambda. \tag{3.1.9}$$

Let $\lambda < 1$. We take $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$. For such ε, there is $g_\varepsilon \in G$ such that $q_r(f - g_\varepsilon) < \lambda + \varepsilon < 1$. This implies the inequality $d(f, g_\varepsilon) < r$, which contradicts (3.1.7). Let now $\lambda > 1$. Since the metric d_2 has the property (A), then by [6], for the above $r > 0$ and $\lambda > 1$, there exists $\varepsilon > 0$ such that $K_{r+\varepsilon} \subset \lambda K_r$. For such $\varepsilon > 0$, there is $g_\varepsilon \in G$ such that $d(f, g_\varepsilon) < r + \varepsilon$. From this we obtain that $f - g_\varepsilon \in \operatorname{int} K_{r+\varepsilon}$, that is, $q_r(f - g_\varepsilon) < \lambda$, which contradicts the equality (3.1.5). Therefore, $\lambda = 1$ and (3.1.4) is true.

Let us now assume that (3.1.8) and $\varepsilon > 0$ are true. Then there exists g_ε such that $q_r(f - g_\varepsilon) < 1 + \varepsilon/r$. As is known [5], if $r \in [2^{-(n_0+1)}, 2^{-n_0}[$, then

$$q_r = \max_{n \le n_0} \frac{1 - 2^n r}{2^n r} p_n(f).$$

From here we get that when $n \le n_0$,

$$\frac{1 - 2^n r}{2^n r} p_n(f - g_\varepsilon) < 1 + \frac{\varepsilon}{r}$$

and

$$2^{-n} p_n(f - g_\varepsilon) < r p_n(f - g_\varepsilon) + r + \varepsilon.$$

Let us divide the last inequality by $1 + p_n(f - g_\varepsilon)$. We find that for each $n \le n_0$,

$$\frac{2^{-n} p_n(f - g_\varepsilon)}{1 + p_n(f - g_\varepsilon)} < r + \frac{\varepsilon}{1 + p_n(f - g_\varepsilon)} < r + \varepsilon.$$

That's why

$$\max_{n \le n_0} \frac{2^{-n} p_n(f - g_\varepsilon)}{1 + p_n(f - g_\varepsilon)} < r + \varepsilon.$$

Since

$$\sup_{n > n_0} \frac{2^{-n} p_n(f - g_\varepsilon)}{1 + p_n(f - g_\varepsilon)} < 2^{-(n_0+1)} \le r < r + \varepsilon,$$

we have $d(f, g_\varepsilon) < r + \varepsilon$. Therefore, for $\varepsilon > 0$, we find $g_\varepsilon \in G$ such that $d(f, g_\varepsilon) < r + \varepsilon$. From this we obtain the validity of the equality (3.1.5), since it is obvious that $\inf\{d(f, g) :\ g \in G\} \ge r$. Using similar reasoning, one can also prove the equivalence of the equalities (3.1.3) and (3.1.4) for the metric (2.5.2). □

Consider the definition of a spline in the case of a normlike metric given by G. Albinus [5]. Assume that information is generated by continuous on F_1 linear functionals. Then $\mathrm{Ker}\, I$ is closed in F_1 and the distance $d(f, \mathrm{Ker}\, I) = r > 0$. From the properties of this metric it follows that $\inf\{q_r(f-h);\ h \in \mathrm{Ker}\, I\} = 1$ [5]. As is known, q_r is equivalent to some seminorm from the given sequence $\{\|\cdot\|_n\}$. The functional corresponding to this information is continuous with respect to this seminorm. Its kernel is closed, since the distance from f to this kernel with respect to the seminorm is positive, namely 1. If it were not closed, it would be everywhere dense and the distance will be zero. Thus, in the definition of a spline with respect to the metric, we always arrive at such a seminorm with respect to which the functional corresponding to this information is continuous. Thus, in the cases under consideration, the definition of a generalized spline needs the requirement of the existence of the best approximation in $\mathrm{Ker}\, I$ only with respect to the metric.

Consider the case when $F_1 = E$ is a metrizable LCS whose topology is generated by a non-increasing sequence of neighborhoods V_n of zero. Denote the Minkowski functional of V_n by $\|\cdot\|_n$, i.e., $V_n = \{f \in E:\ \|f\|_n \le 1\}$. Let X_n be the normed space $X_n = (E/\mathrm{Ker}\,\|\cdot\|_n\ ,\ \widehat{\|\cdot\|_n})$, where $\widehat{\|\cdot\|_n}$ is the associated norm. If instead of F we consider the set V_n for each $n \in \mathbb{N}$, then the canonical maps $K_n : F_1 \to X_n$ will be analogies of the operator $T : E \to X$ and $V_n = \{f \in E:\ \|\widehat{K_n(f)}\|_n \le 1\}$.

Let F_1 be an LCS with a non-increasing sequence of neighborhoods of zero $\{V_n\}$, i.e. $\{V_n\}$ generates a metrizable topology on F_1. This topology can be metrized using the translation-invariant metric d and with absolutely convex balls $K_r = \{x \in F_1;\ d(x,0) \le r\}$. We denote the resulting linear metric space by (F_1, d). In what follows, $q_r(\cdot)$ denotes the Minkowski functionals of the balls K_r and $|\cdot|$ denotes the quasinorm of the metric d.

For metrics (2.5.2) and (2.5.4), the equalities $d(f,G) = d(f,h^*) = r$ and $\inf\{q_r(f-h);\ h \in G\} = q_r(f-h^*) = 1$ are equivalent for an arbitrary closed subset G of the space F_1, i.e. for these metrics, the concepts of proximality and strong proximality coincide.

Let us now consider this problem in the case of the metric (2.5.8).

Proposition 3.1.4. *Let $(E, \mathfrak{T})$ be a Fréchet space with a sequence of seminorms $\{p_n\}$, where $p_1 \ne 0$ and $2p_n \le p_{n+1}$ $(n \in \mathbb{N})$. Let also $G \subset E$ be its closed convex subset. Then for $f \in E \backslash G$ and $r \in I_n$, where I_n is defined according to* (2.5.7)*, the equality* $\inf\{q_r(f-g);\ g \in G\} = 1$ *implies the equality* $d(f,G) = \inf\{d(f,g);\ g \in G\} = r$*. If, in addition, $d(f,G) = r \in \mathrm{int}\ I_n$, then the converse is also true.*

Proof. Let for some $r \in [2^{-n}, 2^{-n+1}[$ we have

$$\inf\{q_r(f-g;\ g \in G\} = \inf\{\lambda \in R^+;\ g \in G \quad \text{and} \quad \lambda^{-1}(f-g) \in K_r\} = 1.$$

Then for an arbitrary number $\varepsilon > 0$, there exist $g_\varepsilon \in G$ and $\lambda_\varepsilon \in [1, 1+\varepsilon[$ such that $\lambda_\varepsilon^{-1}(f-g) \in 2^n r V_{n+1}$. Further, for $s_\varepsilon = r\lambda_\varepsilon$, we obtain that $s_\varepsilon \in [r, r+\varepsilon r[$ and $f - g_\varepsilon \in 2^n r \lambda_\varepsilon V_{n+1} = 2^n s_\varepsilon V_{n+1}$. Moreover, for arbitrary $\varepsilon > 0$, we can assume that $s_\varepsilon \in [r, 2^{-n+1}[$ and, therefore, $f - g_\varepsilon \in K_{s_\varepsilon}$. This means that the inequalities

$$\begin{aligned} d(f,G) &= \inf\{s \in R^+;\ g \in G \text{ and } f-g \in K_s\} \\ &\le \inf\{s \in [2^{-n}, 2^{-n+1}[;\ g \in G \text{ and } f-g \in K_s\} \le r \end{aligned}$$

are valid. If we assume that $d(f,G) < r$, then we immediately obtain a contradiction. Therefore, $d(f,G) = r$.

Let us now prove the converse statement. Let

$$d(f,G) = \inf\{s \in R^+;\ g \in G \text{ and } f-g \in K_s\} = r \in]2^{-n}, 2^{-n+1}[\quad (n \in \mathbb{N}).$$

Then from the condition we get

$$\begin{aligned} &\inf\{s \in [2^{-n}, 2^{-n+1}[;\ g \in G \text{ and } f-g_\varepsilon \in 2^n s V_{n+1}\} \\ &\qquad = r \in]2^{-n}, 2^{-n+1}[\ \ (n \in \mathbb{N}). \end{aligned}$$

This means that for an arbitrary $\varepsilon > 0$, there exist $s_\varepsilon \in [r, r_\varepsilon[$, where $r_\varepsilon = \min(r+\varepsilon, 2^{-n+1})$, and $g_\varepsilon \in G$ such that $f - g_\varepsilon \in K_{s_\varepsilon} = 2^n s_\varepsilon V_{n+1}$. Therefore, for $\lambda_\varepsilon = \frac{s_\varepsilon}{r} \in [1, 1+\frac{\varepsilon}{r}[$ we get

$$f - g_\varepsilon \in 2^n \lambda_\varepsilon r V_{n+1}, \ \text{ i.e. } \ \lambda_\varepsilon^{-1}(f-g_\varepsilon) \in 2^n r V_{n+1} = K_r.$$

Thus, there is an inequality

$$\inf\{q_r(f-g);\ g \in G\} = \inf\{\lambda \in R^+;\ g \in G,\ \lambda^{-1}(f-g) \in K_r\} \le 1.$$

Since $r \ne 2^{-n}$, the strict inequality is excluded and our statement is proved. The case $r \in [1, \infty[$ is treated similarly. □

Corollary. *Let $(E, \mathfrak{T})$ be a metrizable LCS with the metric* (2.5.8)*, $G \subset E$ be its closed convex subset and $f \in E \backslash G$. $g_0 \in G$ is an element of the best approximations of f in G with respect to the metric and $d(f, g_0) = r \in]2^{-n}, 2^{-n+1}[$ $(n \in \mathbb{N})$ (resp. $d(f, g_0) = r > 1$) if and only if $\inf\{p_{n+1}(f-g);\ g \in G\} = p_{n+1}(f-g_0) = 2^n r$ $(n \in \mathbb{N})$ (resp. $\inf\{p_1(f-g; g \in G\} = p_1(f-g_0) = r > 1$).*

It should be noted that if $d(f,G) = r = 2^{-n+1}(n \in \mathbb{N})$, then $\inf\{q_r(f - g); g \in G\} \leq 1$. Indeed, if we assume that $\inf\{q_r(f-g); g \in G\} = \lambda > 1$, then there exists $\varepsilon > 0$ such that $K_{r+\varepsilon} \subset \lambda K_r$. On the other hand, for the specified $\varepsilon > 0$, there is an element $g_\varepsilon \in G$ such that $d(f, g_\varepsilon) < r + \varepsilon$, i.e. $f - g_\varepsilon \in \operatorname{int} K_{r+\varepsilon} \subset \operatorname{int} \lambda K_r$. So, $q_r(f - g_\varepsilon) < \lambda$. This contradicts our assumption. Therefore, $\lambda \leq 1$.

If a spline exists and is unique, then the spline algorithm $\varphi^s : I(F_1) = R^m \to G$ is formally written also as the equality (1.2.2):

$$\varphi^s(y) = S(\sigma(y)), \quad y \in I(F_1)\,. \tag{3.1.10}$$

3.2 Existence of splines in Fréchet spaces

Let us now turn to the question of the existence of splines in the case when F_1 is a metrizable LCS. Let us note first of all that in the case of non-adaptive information I, a spline exists if and only if the subspace $\operatorname{Ker} I$ is strongly proximal in F_1 with respect to the metric. As noted in Section 3.1, in the case of normlike metrics (2.5.2) and (2.5.4), the strong proximality coincides with the ordinary proximality.

This section explores the approximation properties of some classes of subspaces of Fréchet spaces. The well-known theorems of James and Bishop-Phelps for the case of Fréchet spaces are generalized. We also obtain the conditions for the existence of interpolary splines for a non-adaptive information of any cardinality $m \in \mathbb{N}$ (the case $m = 1$ is considered separately).

3.2.1 Generalization of James' Theorem for Fréchet spaces. Condition for the existence of a spline in the case of information of cardinality 1

It is well known ([74] and Theorem 1.3.1) that for a Banach space E, the following statements are equivalent:

a) the space E is reflexive;

b) every linear continuous functional $x' \in E'$ attains its norm on the unit ball of the space E;

c) every closed hypersubspace (i.e. a subspace of the codimension 1) of the space E is proximal;

d) the space E has the proximality property, i.e. every closed subspace of E is proximal;

e) restriction of any linear continuous functional $x' \in E'$ on each closed subspace attains its norm on the unit ball of this subspace;

f) in the space $E = F_1$ with the unit ball F, there exists an interpolation spline for non-adaptive information of any cardinality.

The study of the question whether these properties hold for the Fréchet spaces is much more closely related to the study of topological and geometrical properties of these spaces. To this topic the following works are devoted: [3–5, 53, 185, 193, 198]. Namely, in [53], it was shown that the famous James's Theorem is no longer valid for Fréchet spaces. More precisely, an example of a reflexive, but not totally reflexive space of the Fréchet-Montel type was built, in which for any normlike metric there are non-proximal closed hypersubspace. In [185], it was proved that in Fréchet spaces from the proximality of all closed hypersubspaces, generally speaking, does not follow the proximality of all non-normed closed subspaces. In [3] (see also [5] and [4]), it was proved that the Fréchet nuclear space of all number sequences $\omega = R^N$ (C^N) has the proximality property.

The proximality of closed hypersubspaces in Fréchet spaces under approximation by normlike metrics has been studied in [53, 185, 193]. We will present here the necessary and sufficient condition for proximality of all closed hypersubspaces with respect to normlike metrics (2.5.2), (2.5.4) and metric (2.5.8).

Theorem 3.2.1. *Let* $(E, \mathfrak{T})$ *be a Fréchet space with an increasing sequence of seminorms* $\{p_n\}$ *and a normlike metric* d_1 (*resp.* d_2) *given by the formula* (2.5.2) (*resp.* (2.5.4)). *Then the following statements are equivalent:*

a) *every linear continuous functional* $x' \in K_r^{(1)o}$ (*resp.* $K_r^{(2)o}$) *attains on* $K_r^{(1)}$ (*resp.* $K_r^{(2)}$) *its upper bound, where* $K_r^{(1)}$ $(K_r^{(2)})$ *is the ball of the metric* d_1 (*resp.* d_2).

b) *every closed hypersubspace is proximal in* $(E, \mathfrak{T})$ *with respect to the metric* d_1 (*resp.* d_2).

c) *the space* $(E, \mathfrak{T})$ *is a reflexive quojection and* $(E, \mathfrak{T}) = s\cdot\lim\limits_{\leftarrow}(E/\operatorname{Ker} p_n, \widehat{p}_n)$.

d) *the space* $E'_\beta = s \cdot \lim\limits_{\rightarrow} \operatorname{Ker} p_n^{\perp}$ *is a reflexive strong* (LB)*-space.*

e) *every closed hyperspace is strongly proximal in* $(E, \mathfrak{T})$ *with respect to the metric* d_1 (*resp.* d_2).

Proof. a) $\Rightarrow$ b). Let H be a closed hypersubspace in $(E, \mathfrak{T})$, $f \in E\backslash H$ and $d_1(f, H) = r$. It is known (Lemma 3.1.3) that in this case the equality

$$\inf\{q_r^{(1)}(f - h);\ h \in H\} = 1$$

is valid, where $q_r^{(1)}$ is the Minkowski functional for $K_r^{(1)}$. Then for the associated norm $\widehat{q}_r^{(1)}$ on the quotient space $E/\operatorname{Ker} q_r$ the following equality is also true:

$$\inf\{\widehat{q}_r^{(1)}(\pi_r f - \pi_r h);\ h \in H\} = 1,$$

where $\pi_r : E \to E/\operatorname{Ker} q_r^{(1)}$ is the canonical mapping of E to $E_r = (E/\operatorname{Ker} q_r^{(1)}, \widehat{q}_r^{(1)})$.

Due to the well-known result on the characterization of the element of the best approximation [190], there exists $F' \in E_r'$ such that

$$\sup\{|\langle \pi_r z,\, F'\rangle|;\ z \in K_r^{(1)}\} = 1;$$
$$\langle \pi_r h, F'\rangle = 0,\ \text{when } h \in H,$$
$$|\langle \pi_r f,\, F'\rangle| = 1.$$

Then the equality $F = F' \circ \pi_r$ defines a linear continuous functional on $(E, \mathfrak{T})$ satisfying the following conditions:

$$\sup\{|\langle z,\ F\rangle|;\ z \in K_r^{(1)}\} = 1,$$
$$|\langle f,\ F\rangle| = 1,$$
$$\langle h,\ F\rangle = 0,\ \ \text{when } h \in H.$$

Since $F \in K_r^{(1)o}$, by condition, F attains its upper bound, i.e. there exist $\alpha_0 \neq 0$ and $h_0 \in H$ such that

$$|\langle \alpha_0 f + h_0,\ F\rangle| = q_r^{(1)}(\alpha f + h_0) = 1.$$

Obviously, then $|\alpha_0| = 1$ and, therefore,

$$\Big|\Big\langle f + \frac{h_0}{\alpha_0},\ F\Big\rangle\Big| = 1 = q_r^{(1)}\Big(f + \frac{h_0}{\alpha_0}\Big).$$

In this case, $d_1(f, H) = d_1(f, -\frac{h_0}{\alpha_0})$, i.e. for $f \in E$ in H there is an element of best approximation. This means that the hypersubspace H is proximal in $(E, \mathfrak{T})$ with respect to d_1.

b) $\Rightarrow$ c). Let us first prove that under the conditions of statement b) for any normlike metric d the quotient spaces $E/\operatorname{Ker} q_n$ are reflexive Banach spaces according to the norms $\widehat{q}_r$. To do this, we will show that every closed in the normed space $(E/\operatorname{Ker} q_n,\, \widehat{q}_r)$ hypersubspace is proximal with respect to the norm $\widehat{q}_r$. This follows from the fact that, by virtue of the well-known James's Theorem, the linear continuous functional $F' \in (E/\operatorname{Ker} q_n,\, \widehat{q}_r)'$ attains its upper bound on the unit ball $\pi_r K_r$ of the space $(E/\operatorname{Ker} q_n,\, \widehat{q}_r)$ if and only if $H_r = \{\pi_r h \in \pi_r(H); \langle \pi_r h,\, F'\rangle = 0\}$ is proximal in $(E/\operatorname{Ker} q_n,\, \widehat{q}_r)$. But $F' \in (E/\operatorname{Ker} q_n,\, \widehat{q}_r)'$ attains its upper bound on the unit ball $\pi_r K_r$ of space $(E/\operatorname{Ker} q_n,\, \widehat{q}_r)$ if and only if $F = F' \circ \pi_r$ attains the upper bound on K_r. On the other hand, due to [53], from the proximality of all closed hypersubspaces we obtain that every linear continuous functional $x' \in K_r^0$ attains its upper bound. If we now apply the normability condition from [75] to the case of Fréchet space $E/\operatorname{Ker} q_n$ and neighborhood in the quotient topology $\pi_r K_r$, we obtain that $(E/\operatorname{Ker} q_n,\, \widehat{q_r})$ is a reflexive Banach space.

Obviously, the proved statement is also true for the metric d_1. Indeed, the metric d_1 has the property (B) with respect to the sequence of seminorms $\{p_n\}$. In the case of the metric d_1, this means that for $r \in [1, \infty[$, the seminorms $q_r^{(1)}$ are equivalent to p_1, and for $r \in [2^{-n},\, 2^{-n+1}]$, the seminorms $q_r^{(1)}$ are equivalent to p_{n+1} $(n \in N)$. Therefore, the quotient space $E/\operatorname{Ker} p_n$ is reflexive Banach space according to the norm $\widehat{p}_n$ for each $n \in \mathbb{N}$. Therefore, $(E, \mathfrak{T}) = s \cdot \lim\limits_{\leftarrow}(E/\operatorname{Ker} p_n,\, \widehat{p}_n)$ is a reflexive quojection.

$c) \Leftrightarrow d)$ follows from Corollary 2 of Theorem 2.3.2.

c) $\Rightarrow$ a). The proximality of all closed hypersubspaces with respect to metric d_1 follows from ([185], p. 144, Theorem 1). Likewise, this theorem is also proven for the metric d_2.

c) $\Rightarrow$ e) follows from Lemma 3.1.3. □

Corollary. *In the space $L^p_{loc}(\Omega)$ $(1 < p < \infty)$, where Ω is an open domain in R^l, all closed hypersubspaces are strongly proximal with respect to the normlike metrics d_1 and d_2.*

Indeed, it is easy to check that if $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$, $\Omega_n \subset \operatorname{int}\, \Omega_{n+1}$ $(n \in \mathbb{N})$ and

$$p_n(f) = \left(\int\limits_{\Omega_n} |f(t)|^p dt\right)^{1/p},$$

then the quotient space $L^p_{loc}(\Omega)/\operatorname{Ker} p_n$ is isomorphic to the Banach space $L^p(\Omega_n)$ which is the space of p-summable functions on Ω_n.

Theorem 3.2.2. *Let $(E, \mathfrak{T})$ be a Fréchet space with the sequence of seminorms $\{p_n\}$, where $p_1 \neq 0$ and $2p_n \leq p_{n+1}$ $(n \in \mathbb{N})$. Then the following statements are equivalent:*

a) *each closed hypersubspace is proximal in E with respect to the metric d defined by the formula* (2.5.8)*;*

b) *each continuous linear functional $x' \in K_r^0$ $(r > 0)$ attains its supremum on K_s $(s \leq r)$, where K_r^0 is the polar of K_r in the dual space E';*

c) *the space $(E, \mathfrak{T})$ is a reflexive quojection and $(E, \mathfrak{T}) = s\cdot\lim\limits_{\leftarrow}(E/\operatorname{Ker} p_n, \widehat{p}_n)$;*

d) *the space $E'_\beta = s \cdot \lim\limits_{\rightarrow} \operatorname{Ker}\, p_n^{\perp}$ is a reflexive strict (LB)-space;*

e) *each closed hypersubspace is strongly proximal in $(E, \mathfrak{T})$ with respect to the metric d defined by the formula* (2.5.8).

Proof. a) $\Rightarrow$ b). Let $x' \in K_r^0$ $(r > 0)$, $s \leq r$ be some positive number, $H = \{x \in E; \langle x, x' \rangle = 0\}$ and $x_0 \in E$ such that $\langle x_0, x' \rangle \neq 0$. It follows from this that $q_r(x_0) \neq 0$ and therefore $q_s(x_0) \neq 0$. Next, from the condition we obtain that $0 < l = \inf\{q_s(x_0 - h); h \in H\}$. Without loss of generality, we can assume that $l = 1$. By virtue of Proposition 3.1.4 and by condition, for some $h_0 \in H$ we will have $d(x_0, H) = d(x_0, h_0) = s$. Therefore, $q_s(x_0 - h_0) = \inf\{q_s(x_0 - h); h \in H)\} = 1$. From the well-known result on the characterization of the element of the best approximation [190], we obtain that there exists a functional $F \in K_s^0$ satisfying the conditions $\langle x_0, F \rangle = 1$ and $\langle h, F \rangle = 0$, when $h \in H$, i.e. $F = \lambda x'$ for some λ. It follows that x' also attains on K_s its upper bound.

b) $\Rightarrow$ c) From the condition we obtain that every linear continuous functional $x' \in E'_{K_r^0}$ $(r \in I_n)$, where $E'_{K_r^0}$ is a Banach space, spanned by K_r^0, attains its upper bound on K_r. But, as is known (see Section 1.4), the Banach spaces $E'_{K_r^0}$ and $(E/\operatorname{Ker} q_n, \widehat{q}_r)'$ are isometric and this isometry is realized by restriction on the space $(E/\operatorname{Ker} q_n, \widehat{q}_r)'$ of adjoint algebraic isomorphism $k_n' : (E/\operatorname{Ker} q_n, \widehat{q}_r)' \to \operatorname{Ker} q_r^{\perp} \subset E'$. Therefore, the following equalities hold:

$$\langle k_n x, F \rangle = \langle x, k_n' F \rangle = \langle x, x' \rangle, \quad x \in E, \quad F \in (E/\operatorname{Ker} q_n, \widehat{q}_r)'.$$

It follows from here that $x' \in E'_{K_r^0}$ attains its upper bound at K_r if and only if $F' = k_n^{'(-1)} x'$ attains its upper bound on $k_n(K_r) = \{k_n x \in E/\operatorname{Ker} q_n;\ \widehat{q}_r(k_n x) \leq 1\}$, i.e. on the unit ball of the space $(E/\operatorname{Ker} q_n; \widehat{q}_r)$. If now apply the normability condition from [75] to the Fréchet space $E/\operatorname{Ker} q_n$ and the norm $\widehat{q}_r$, then we obtain that the quotient space $E/\operatorname{Ker} q_n$ is a reflexive Banach space according to the norm $\widehat{q}_r$. It remains to note that for $r \in I_n$, the space $(E/\operatorname{Ker} q_n, \widehat{q}_r)$ is isomorphic to the space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$. Therefore, $(E, \mathfrak{T}) = s \cdot \varprojlim (E/\operatorname{Ker} p_n, \widehat{p}_n)$ is reflexive quojection.

c) $\Leftrightarrow$ d) was already proved in the proof of Theorem 3.2.1.

c) $\Rightarrow$ a) Let H be a closed hypersubspace of the space E, $f \in E \backslash H$ and $d(f, H) = r \in I_n$. Then, as was proved above, we have

$$\inf\{q_r(f - h); h \in H\} = \lambda \leq 1.$$

If $\lambda \neq 0$, then we get that $k_n(H)$ is closed hypersubspace in a reflexive Banach space $(E/\operatorname{Ker} q_n, \widehat{q}_r)$. Therefore, there exists $h_0 \in H$ such that

$$\inf\{\widehat{q}_r(k_n(f - h); h \in H\} = \widehat{q}_r(k_n(f - h_0)) = \lambda.$$

Further, if $r \in \operatorname{int} I_n$, then $\lambda = 1$ and $d(f, H) = d(f, h_0) = r$. If $r = 2^{-n+1}$ and $0 \leq \lambda < 1$, then for some $h_1 \in H$ we obtain that $f - h_1 \in \operatorname{int} V_n$. Let us assume that $f - h_1 \in 2\,\mathrm{int}\, V_{n+1}$, then we get $d(f, h_1) \leq 2^{-n} p_{n+1}(f - h_1) < r$, which is impossible. This means that $f - h_1 \in \mathrm{int}\, V_n \backslash 2\,\mathrm{int}\, V_{n+1}$ and $d(f, h_1) = r = 2^{-n+1}$, i.e. h_1 is the best approximation of f in H with respect to the metric d.

e) $\Rightarrow$ a) obviously.

c) $\Rightarrow$ e). Let again H be a closed hyperspace of E, $f \in E \setminus H$ and $d(f, H) = r \in I_n$. When proving implication c) $\Rightarrow$ a), it was proved that

$$\inf\{\widehat{q}_r(K_n(f - h);\ h \in H\} = \lambda \leq 1.$$

Let $\{K_n h_m\}$ be a minimizing sequence, i.e.

$$\inf\{\widehat{q}_r(K_n f - K_n h);\ h \in H\} = \lim_{m \to \infty} \widehat{q}_r(K_n f - K_n h_m) = \lambda.$$

Then the sequence $\{K_n h_m\}$ is bounded in the reflexive Banach space $(E/\,\mathrm{Ker}\, p_n, \widehat{p}_n)$ and therefore some of its subsequences converges to the element $K_n h_0 \in K_n(H)$. If $\lambda = 0$, then $\widehat{p}_n(K_n f - K_n h_0) = 0$ and $f - h \in \mathrm{Ker}\, p_n$. The element h_0 is a strong best approximation for f in H, which is proved similarly to the proof of implication c) $\Rightarrow$ a). $\square$

Corollary. *The Fréchet space $(E, \mathfrak{T})$ with the generating sequence of absolutely convex neighborhoods $\{V_n\}$ is reflexive quojection if and only if every linear continuous functional $x' \in V_n^0$ $(n \in \mathbb{N})$ attains its upper bound on V_n.*

This corollary is a generalization of James's Theorem.

Applying of these results, we obtain conditions for the existence of interpolation splines.

Theorem 3.2.3. *Let F_1 be a Fréchet space and F be a closed absolutely convex neighborhood of zero of the space F_1. For each $y \in R$ and non-adaptive information $I(f) = L(f)$ of cardinality 1, an interpolation spline exists if and only if the space (F_1, μ_F), where μ_F is the Minkowski functional for F, is isomorphic to the reflexive Banach space.*

Proof. From the conditions of the theorem it follows that μ_F is a continuous seminorm on F_1. If we assume that μ_F is the norm on F_1, then from conditions for the existence of an interpolation spline for each $y \in R$ and $L \in F_1'$ we obtain that every linear and bounded on F functional attains its upper bound on it. Now, applying ([75], Theorem 1), we find that (F_1, μ_F) is a reflexive Banach space. Let us say now that μ_F is a continuous seminorm on F_1. Let us consider the subspace $\mathrm{Ker}\, \mu_F$ and the quotient space $F_1/\,\mathrm{Ker}\, \mu_F$ with the associated norm $\widehat{\mu}_F$ defined by the equality $\widehat{\mu}_F(kx) = \mu_F(x)$, where $k : F_1 \to F_1/\,\mathrm{Ker}\, \mu_F$ is a canonical mapping. The normed space $(F_1/\,\mathrm{Ker}\, \mu_F, \widehat{\mu}_F)$ is something other than the space

$(X, \|\cdot\|)$ from [158]. From the definition of the norm $\widehat{\mu}_F$ and from the existence of an interpolation spline in F_1 for arbitrary $y \in R$ and information $I(f) = L(f)$, we obtain that the interpolation spline also exists in the space $(F_1/\operatorname{Ker}\mu_F, \widehat{\mu}_F)$ for arbitrary $y \in R$ and information $\widehat{I}(f) = \widehat{L}(kf)$, where $\widehat{L}(kf) = L(f)$ for each $f \in F_1$. But the quotient space $F_1/\operatorname{Ker}\mu_F$ is itself Fréchet space. If we again apply ([75], Theorem 1) for the space $F_1/\operatorname{Ker}\mu_F$ and the norms of $\widehat{\mu}_F$, we obtain that the space $(F_1/\operatorname{Ker}\mu_F, \widehat{\mu}_F)$ is isomorphic to a reflexive Banach space. It remains to note that the spaces $(F_1/\operatorname{Ker}\mu_F, \widehat{\mu}_F)$ and (F_1, μ_F) are isomorphic.

The converse statement can be easily obtained from Theorem 1.3.1, since $\operatorname{Ker} I = \operatorname{Ker} L$ and it is either dense or closed. In the closedness case that takes place in our case, due to the reflexivity of the space $(F_1/\operatorname{Ker}\mu_F, \widehat{\mu}_F)$, the subspace $\operatorname{Ker} L$ is proximal with respect to $\widehat{\mu}_F$. □

Theorem 3.2.4. *Let F_1 be a Fréchet space generating with a sequence of non-increasing absolutely convex neighborhoods of zero $\{V_n\}$. Then the following statements are equivalent:*

a) *F_1 is a reflexive quojection.*

b) *For each $y \in R$ and non-adaptive information $I(f) = L(f)$ of cardinality 1, an interpolation spline exists for every absolutely convex neighborhood V_n of the space F_1.*

c) *$V_n = B_n + \operatorname{Ker}\mu_{V_n}$, where B_n is absolutely convex and weakly compact set in F_1.*

Proof. a) ⇒ b). According to the theorem, the quotient spaces $(F_1/\operatorname{Ker}\mu_{V_n}, \widehat{\mu}_{V_n})$ are reflexive Banach spaces for each $n \in \mathbb{N}$. From Theorem 3.2.2 we immediately obtain the validity of statement b).

b) ⇒ c). From Theorem 3.2.2 it follows that the spaces $(F_1/\operatorname{Ker}\mu_{V_n}, \widehat{\mu}_{V_n})$ are reflexive Banach spaces for each $n \in \mathbb{N}$. From Corollary 1 of Theorem 2.3.2 we obtain that the strong dual space $(F_1/\operatorname{Ker}\mu_{V_n}, \widehat{\mu}_{V_n})'$ is isomorphic to the Banach space $\operatorname{Ker}\mu_{V_n}^{\perp}$, which is considered in the induced topology of strongly conjugate space. This topology is generated by polar of the set V_n^0 (see also the proof of Corollary 1 of Theorem 2.3.1). That's why there are bounded sets B_n $(n \in \mathbb{N})$ in the space F_1 such that $V_n^0 = B_n^0 \cap \operatorname{Ker}\mu_v^{\perp}$. If in this equality we go to polar in the space F_1, then, due to the reflexivity of the space F_1, we obtain the equality $V_n = B_n + \operatorname{Ker}\mu_{V_n}$.

c) ⇒ a) is proved similarly to the proof of implications c) ⇒ a) of Theorems 2.3.2. □

It should also be noted that, by virtue of Corollary 2 of Theorem 2.3.2, in the statement c) of Theorem 3.2.4, the set B_n can be chosen to be independent of $n \in \mathbb{N}$.

Theorem 3.2.5. *Let F_1 be a Fréchet space with a generating sequence of non-increasing absolutely convex neighborhoods of zero $\{V_n\}$, with normlike metrics* (2.5.2)*,* (2.5.4) *and metric* (2.5.12)*. For any $y \in R$ and non-adaptive information $I(f) = L(f)$ of cardinality* 1*, the spline exists if and only if the space F_1 is reflexive quojection.*

The proof of this theorem actually follows from the proofs of Theorems 2.3.1 and 2.3.2. Consequently, a generalization of James's Theorem to the case of Fréchet spaces gives a necessary and sufficient condition for the existence of a spline in the case of non-adaptive information of cardinality 1.

3.2.2 Reflexive Fréchet spaces with non-proximal hypersubspaces

Let us give examples of reflexive Fréchet spaces for which there is non-adaptive information $I(f) = L(f)$ of cardinality 1 such that the spline does not exist for some $y_0 \in R$.

Let us first point out on the projective limit of reflexive Banach spaces, which has a non-proximal hypersubspace with respect to metrics d_1 and d_2. Consider the non-normable projective limit of reflexive Banach spaces $(E, \mathfrak{T})$ with a non-increasing generating sequence of strictly convex norms $\{\|\cdot\|_n\}$. In [189], it was proved that in this case the norms $q_r^{(2)}$, $r \in]0, 1/2[$, are also strictly convex. If all closed in $(E, \mathfrak{T})$ hypersubspaces are proximal with respect to the metric d_2, then, due to [185], the space $(E, \mathfrak{T})$ turns out to be normable with respect to some norm $\|\cdot\|_{n_0}$, $n_0 \in \mathbb{N}$, which is impossible. A similar result is valid for the metric d_1.

In particular, this property is possessed by the countably normed space $W^{p,\infty}(R)$, which was introduced in [189]. $W^{p,\infty}(R)$ is the space of all functions having generalized derivatives of all orders such that $f^{(s)} \in L^p(R)$. For the space $W^{p,\infty}(R)$, the representation $W^{p,\infty}(R) = \bigcap\limits_{n=0}^{\infty} W_p^n(R)$ is true, where $W_p^n(R)$ is the Sobolev space for $n \in \mathbb{N}$. The topology of this space is determined using the sequences of strictly convex norms

$$\|f\|_{p,n} = \|f\|_p + \cdots \|f^{(n-1)}\|_p, \quad n \in \mathbb{N},$$

where

$$\|f\|_p = \left(\int\limits_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}.$$

Embedding theorems for this space in Sobolev spaces can be found in Section 2.6.

From the reasoning given in Section 3.2, it also follows that with respect to d_1 and d_2 non-proximal closed hypersubspaces have:

1. Reflexive Fréchet spaces in which total bounded sets do not exist. This follows from the fact that in its strong dual space there are no continuous norms.

2. Reflexive Fréchet spaces on which there exist continuous norms.

3. Spaces of Fréchet-Montel type, non-isomorphic to the space ω.

Now we select a class of projective limits of reflexive Banach spaces in which for any normlike metric d there exist nonproximal hypersubspaces. Let $(E,\mathfrak{T})$ be a nuclear Fréchet space, non-isomorphic to ω, with an arbitrary normlike metric d. It is well known that each nuclear space is represented in the form of the projective limit of a sequence of Hilbert spaces. On the other hand, if in $(E,\mathfrak{T})$ all closed hypersubspaces are proximal with respect to the normlike metric d, then as was established in the proof of implication b $\Rightarrow$ c) of Theorem 3.2.1, quotient spaces $E/\operatorname{Ker} q_n$ are Banach in the associated norm $\widehat{q}_r$. But, since the quotient spaces of nuclear spaces are nuclear, $E/\operatorname{Ker} q_n$ will be finite-dimensional and $(E,\mathfrak{T})$ will turn out to be isomorphic space ω, which is impossible. In particular, the space ω is not isomorphic to the nuclear space of all infinitely differentiable functions on the line $\mathcal{E}(R)$.

There is an example of a reflexive Fréchet space F_1 (even space (FM)), quotient space $F_1/\operatorname{Ker}\mu_F$ of which is not complete with respect to the norm $\widehat{\mu}_F$. Therefore, in such spaces there will always be absolutely convex neighborhoods F and non-adaptive information of cardinality 1, for which there are no interpolation splines. Similar examples of reflexive Fréchet spaces are specified in Section 3.2. Moreover, if F_1 is a nuclear Fréchet space, non-isomorphic to the space $\omega = R^N$, then for any neighborhood F of zero there is a non-adaptive information of cardinality 1, for which interpolation splines do not exist.

3.3 Generalization of the Bishop–Phelps theorem for quojections

Here we generalize the well-known Bishop-Phelps theorem into an approximate form for the metric (2.5.8). In particular, simple characterizations of proximal hypersubspaces in function spaces $C(T)$, $L^1_{loc}(R)$ are given, where T is a separable locally compact space, countable at infinity. In the works [19, 122], the following theorem was proved.

Theorem 3.3.1. *Let $(E,\mathfrak{T})$ be a quejection with a non-increasing generating sequence of closed neighborhoods of zero $V_n = 2^{-n+1}B + \operatorname{Ker} p_n$, where B is a closed, bounded absolutely convex subset in E and p_n are the Minkowski functionals for V_n. Then the following statements are valid:*

a) *The set $\mathbb{P}$ of all functionals $x' \in E'$ attaining its upper bound on each V_k $(k \geq n)$ as soon as $x' \in E'_{V_n^0}$ ($E'_{V_n^0}$ is the Banach space spanned on the polar*

V_n^0 of the neighborhood V_n), is everywhere dense in the strongly conjugate strict (LB)-space E'_β.

b) *Closed hypersubspace $H = \operatorname{Ker} x' = \{h \in E; \langle h, x'\rangle = 0\}$, where $x' \in E'$, is proximal in E with respect to the metric* (2.5.8) *if and only if $x' \in \mathbb{P}$.*

Proof. a) As is known, the quotient space $E/\operatorname{Ker} p_n$ is Banach space according to the associated norm $\widehat{p}_n$. Passing on to polars in E' in the given equality, we obtain that $V_n^0 = 2^{n-1}B^0 \cap \operatorname{Ker} p_n^\perp$, where $\operatorname{Ker} p_n^\perp$ is a weakly closed subspace of E' orthogonal to $\operatorname{Ker} p_n$. It follows that $\operatorname{Ker} p_n^\perp$, considered in the induced topology of the space E'_β, coincides with the Banach space $E'_{V_n^0}$. Therefore, in the future we will consider $\operatorname{Ker} p_n^\perp$ with unit ball V_n^0 and the corresponding norm $\|\cdot\|'_n$, assuming that it is isometric to the space $(E/\operatorname{Ker} p_n, \widehat{p}_n)'$. Note that the indicated isometry is carried out by the mapping $k_n^{'}$, which is conjugate to the mapping k_n. From the above it follows that $x' \in \operatorname{Ker} p_n^\perp$ attains its upper bound on V_n, i.e. is supporting to V_n if and only if $F = k_n^{'(-1)} x'$ is supporting to $\widehat{V}_n = k_n V_n = 2^{-n+1} k_n(B)$. On the other hand, one can directly verified that if $x' \in \operatorname{Ker} p_n^\perp$ is supporting to V_n, then it is supporting to B and, therefore, to each V_k $(k \geq n)$. Denote by $\mathbb{P}_n$ the set of functionals $x' \in \operatorname{Ker} p_n^\perp$ that are supporting to V_n. Then $\mathbb{P}_n \subset \mathbb{P}_{n+1}$ and $\mathbb{P} = \bigcup_{n\in\mathbb{N}} \mathbb{P}_n$. From the Bishop-Phelps Theorem it follows that $k_n^{'(-1)}(\mathbb{P}_n)$ is everywhere dense in $(E/\operatorname{Ker} p_n, \widehat{p}_n)'$ and therefore $\mathbb{P}_n$ is everywhere dense in $(\operatorname{Ker} p_n^\perp, \|\cdot\|'_n)$. Let us now prove that $\mathbb{P}$ is everywhere dense in the space $E'_\beta = s\cdot \lim\limits_{\rightarrow} \operatorname{Ker} p_n^\perp$, i.e. in the strong inductive limit of the sequences of Banach spaces $\{(\operatorname{Ker} p_n^\perp, \|\cdot\|'_n)\}$. Let $x'_0 \in E'$, then $x'_0 \in \operatorname{Ker} p_{n_0}^\perp$ for some $n_0 \in \mathbb{N}$. By Corollary 1 of Proposition 2.2.2, in the strict (LB)-space E'_β there is a neighborhood basis $\mathcal{U}$ such that for each neighborhood of $U \in \mathcal{U}$ its Minkowski functional p_U is the norm on E', inducing on each $\operatorname{Ker} p_n^\perp$ the topology of the norm $\|\cdot\|_n$. Therefore, for any neighborhood $U \in \mathcal{U}$, there exists $a' \in \mathbb{P}_{n_0}$ such that $x'_0 - a' \in U \cap \operatorname{Ker} p_{n_0}^\perp \subset U$.

b) Let $x' \in \mathbb{P}$, $x' \in E'_{V_{n+1}^0} \backslash E'_{V_n^0}$, $f \in E\backslash H$, where $H = \operatorname{Ker} x'$ and $d(f, H) = r \in [2^{-m}, 2^{-m+1}[$. Obviously, then $m \geq n$. By Proposition 3.1.2 we have that $\inf\{q_r(f-h); h \in H\} = \lambda \leq 1$. As noted when proving implication a) $\Rightarrow$ b) of Theorem 3.2.2, there is a linear continuous functional F on E with the following properties: $\sup\{|\langle z, F\rangle|; z \in K_r\} = 1$, $\langle f, F\rangle = \lambda$ and $\langle h, F\rangle = 0$ for all $h \in H$. From these conditions we obtain that $F = \mu\, x'$. Without loss of generality, we can assume that $\mu = 1$. By condition, x' attains its upper bound at every K_r, where $r < 2^{-n+1}$. Therefore, there exist $\alpha_0 \neq 0$ and $h_0 \in H$ such that

$$\langle \alpha_0 f + h_0,\, x'\rangle = 1 = q_r(\alpha_0 f + h_0)$$

and therefore
$$\Big\langle f+\frac{h_0}{\alpha_0},\,x'\Big\rangle=\frac{1}{\alpha_0}=\lambda=q_r\Big(f+\frac{h_0}{\alpha_0}\Big).$$

Let us now consider two cases: $2^{-m}<r<2^{-m+1}$ and $r=2^{-m}$. In the first case we obtain that $\lambda=1$, i.e. $d(f,h_1)=r$ and $h_1=-\frac{h_0}{\lambda_0}$ is the best approximation element for $f\in E$ in H with respect to the metric (2.5.8).

Let now $r=2^{-m}$. If in this case $\lambda=1$, then the statement is proved in a similar way. If $\lambda<1$, then $f-h_1=f+h_0/\alpha_0\in\operatorname{int}V_{m+1}$. Let us assume that $f-h_1\in 2\operatorname{int}V_{m+2}$, then we obtain that $d(f,h_1)=2^{-m-1}p_{m+2}(f-h_1)<2\cdot 2^{-m-1}=2^{-m}$, which is impossible. Hence, $f-h_1\in\operatorname{int}V_{m+1}\setminus\operatorname{int}2\,V_{m+2}$ and $d(f,h_1)=2^{-m}=r$, i.e. h_1 is the best approximation of f in H.

Let us now prove the converse statement. Let $x'\in E'_{V^0_{n+1}}\setminus E'_{V^0_n}$ and $H=\operatorname{Ker}x'$ be proximal in E with respect to the metric (2.5.8). It should be proved that $x'\in\mathbb{P}$, i.e. x' attains its upper bound on each neighborhood V_k $(k\geq n+1)$ and on each K_r $(r<2^{-n+1})$. Let $r\in[2^{-m},2^{-m+1}[$, where $m\geq n$ and $\sup\{|\langle z,x'\rangle|;z\in K_r\}=\delta$. Without loss of generality, we can assume that $\delta=1$. Next, we choose $f_0\in E$ such that $\langle f_0,x'\rangle\neq 0$, then $q_r(f_0)\neq 0$. Therefore, we also have that $\inf\{q_r(f_0-h);h\in H\}=l>0$. Without loss of generality, we can again assume that $l=1$. By virtue of Proposition 3.1.4, we obtain $d(f_0,H)=r$. By assumption, there exists $h_0\in H$ such that $d(f_0,h_0)=r$. Then it is obvious that
$$q_r(f_0-h_0)=\inf\{q_r(f_0-h);h\in H\}=1.$$
By virtue of the applied result, we obtain that there exists $F\in E'$ such that $\sup\{|\langle z,F\rangle|;\ z\in K_r\}=1$, $\langle f_0,F\rangle=1=q_r(f_0-h_0)$ and $\langle h,F\rangle=0$ for all $h\in H$. From this we obtain that $F=\mu\,x'$ and it attains its upper bound on K_r. Therefore, $x'\in\mathbb{P}$. The case $x'\in E'_{V^0_1}$ is considered similarly. □

In [39], we gave a description of the set $\mathbb{P}$ for known quojections, generalizing the results from [214] and [122]. From here we obtain that the following is valid.

Proposition 3.3.2. *Let $T=\bigcup\limits_{n\in\mathbb{N}}T_n$ be a separated locally compact space countable at infinity and $C_R(T)$ be the quojection of all real continuous functions on T with the sequence of seminorms $p_n(f)=2^{n-1}\max\{|f(t)|;\ t\in T_n\}$ and the metric* (2.5.8). *Then the following statements are valid:*

a) *The set $\mathbb{P}\subset C'_R(T)=M_C(T)$ consists of Radon measures μ with compact support for which μ^+ and μ^- have disjoint supports.*

b) *The closed hypersubspace* $\operatorname{Ker}\mu$ *is proximal in $C_R(T)$ with respect to the metric* (2.5.8) *if and only if $\mu\in\mathbb{P}$.*

Proposition 3.3.3. *Let $L^1_{loc}(R)$ be a quojection with the sequence of seminorms*

$$p_n(f) = \int\limits_{-n}^{n} |f(t)|dt$$

and the metric (2.5.8). *Then the following statements are valid.*

a) *The set $\mathbb{P} \subset (L^1_{loc}(R))' = L^\infty_0(R)$ for the space $L^1_{loc}(R)$ consists of all $F \in L^\infty_0(R)$ attaining its essential supremum on a set of positive measure, where $L^\infty_0(R)$ is the space of all (equivalent classes) measurable and essentially bounded real functions on R equal to zero outside of some compact interval.*

b) *The closed hypersubspace* $\operatorname{Ker} F$ *is proximal in $L^1_{loc}(R)$ with respect to the metric* (2.5.8) *if and only if $F \in \mathbb{P}$.*

The Bishop–Phelps Theorem shows that in the case of quojections the set $\mathbb{P}$ of all non-adaptive information $I(f) = L(f)$ of cardinality 1, for which a spline exists for any $y \in R$, is everywhere dense in the strong topology of the dual strict (LB)-space.

3.4 Sufficient conditions for the existence of splines of any cardinality

It is well known that every Fréchet space $(E, \mathfrak{T})$ is isomorphic to a closed subspace of the product of the sequence of Banach spaces, i.e. to trivial quojection. Therefore, the closed subspace of a quojection, generally speaking, is not even distinguished. Let G be a distinguished closed subspace of the quojection E. From the characterization of the strong dual to the subspace [65], we obtain that if G is a quojection, then the quotient space $E'_\beta/G^\perp$ of the strict (LB)-space E'_β is a strict (LB)-space.

Therefore, by virtue of Theorem 2.3.3, we have that every closed subspace of reflexive quojections $(E, \mathfrak{T})$ is quojection if and only if each quotient space of strictly (LB)-space E'_β is a strict (LB)-space. By [16], if the Fréchet space $(E, \mathfrak{T})$ is not isomorphic to $B \times \omega$, where B is is a Banach space, then it has a nuclear Koethe subspace. On the another hand, each closed subspace G of the space $B \times \omega$ is either of the same shape, or is isomorphic to a Banach space, or isomorphic to the space ω. It follows that every closed subspace of a quojection E is a quojection if and only if E is isomorphic to the space $B \times \omega$, where B is a Banach space. This, in turn, is equivalent to the fact that each quotient space E'_β over a weakly closed subspace is strict (LB)-space.

Theorem 3.4.1. *Let $(E, \mathfrak{T})$ be a non-normed Fréchet space with the sequence of seminorms $\{p_n\}$, where $p_1 \neq 0$ and $2p_n \leq p_{n+1}$ $(n \in \mathbb{N})$. Then the following statements are equivalent:*

a) *Each nonnormable closed subspace of the space E is proximal with respect to the metric* (2.5.8).

b) *Each continuous linear functional $x' \in K_r^0$ $(r > 0)$ attains its supremum on K_s $(s \le r)$, and its restriction to any nonnormable closed subspace G of the space $(E, \mathfrak{T})$ attains its supremum on $K_s \cap G$.*

c) *For the space $(E, \mathfrak{T})$ and each of its nonnormable closed subspace G, the quotient spaces*

$$(E/\operatorname{Ker} p_n, \widehat{p}_n) \;\; (n \in \mathbb{N}) \quad \textit{and} \quad (G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G}), \;\; n \in \mathbb{N},$$

are reflexive Banach spaces.

d) *For the space $E'_\beta = s \cdot \lim\limits_{\rightarrow}(\operatorname{Ker} p_n^\perp, \|\cdot\|'_n)$, where $\|\cdot\|'_n$ is the Minkowski functional for V_n^0 and for each nonnormed subspace G of the space E, the equality*

$$G'_\beta = s \cdot \lim_{\rightarrow}(\operatorname{Ker} p_n^\perp, \|\cdot\|'_n)/(\operatorname{Ker} p_n^\perp \cap G^\perp)$$

is true.

e) *The quotient spaces $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ $(n \in \mathbb{N})$ are reflexive Banach spaces and for an arbitrary nonnormable closed subspace G, the sets $k_n(G)$ are closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ $(n \in \mathbb{N})$, where $k_n : E \to E/\operatorname{Ker} p_n$ are canonical homomorphisms.*

Proof. a) $\Rightarrow$ b). The first part of b) follows from Theorem 3.2.2. Let G be a nonnormable closed subspace of the space E, $x' \in K_r^0$ and $s \le r$. Let p_m $(m > 1)$ (resp. p_1) be the first seminorm in the sequence $\{p_m\}$, whose restriction to G is not identically zero. Let us show that this is equivalent to the equality $\sup\{|g|;\, g \in G\} = 2^{-m+2}$ $(m > 1)$ (resp. $\sup\{|g|;\, g \in G\} = \infty$). Indeed, if $2^{-m+1} < l < 2^{-m+2}$, then for $g' \in G$ with $p_m(g') = 1$ we have

$$q_l(2^{m-1} l g') = 2^{-m+1} l^{-1} \cdot 2^{m-1} \cdot l\, p_m(g') = 1,$$

i.e. $|2^{m-1} l g'| = l$. Further, if there exists an element $g_0 \in G$ such that $|g_0| > 2^{-m+2}$, then $g_0 \overline{\in} K_{2^{-m+2}}$ and $p_{m-1}(g_0) > 1$, which contradicts the choice of number m. The converse assertion is easily proved, and we omit the proof. Hence $G \subset K_s$ for any $r \ge s \ge 2^{-m+2}$, the restriction of x' to G is identically equal to zero, and x' attains its supremum at the element $0 \in G \cap K_s$.

Assume now that $2^{-m+1} \le s < 2^{-m+2}$ and that $g_1 \in G$ is such that $\langle g_1, x' \rangle \ne 0$. Then $p_m(g_1) \ne 0$. Consider the closed subspace $H_G \subset E$ defined by the equality $H_G = H \cap G$, where $H = \operatorname{Ker} x'$. Obviously, $g_1 \overline{\in} H_G$ and $G = [g_1] + H_G$, i.e., G is the topological sum of the one-dimensional space $[g_1]$ spanned by g_1 and the subspace H_G. It follows by assumption that the closed hypersubspace H is also closed with respect to the seminorms q_s. This also gives us that H_G is closed in G with respect to the restriction of q_s to G, which means that $\inf\{q_s(g_1 - h); h \in H_G\} = l > 0$. We can assume that $l = 1$. By the assumption, this implies that $d(g_1, H_G) = d(g_1, h_0) = s$ for some $h_0 \in H_G$, i.e. $q_s(g_1 - h_0) = 1$. If we now repeat the arguments in the proof of the implications of a) ⇒ b) in Theorem 3.2.2, we get that the restriction of x' to G attains its supremum on $K_s \cap G$. The case when $s \in [2^{-n+1}, 2^{-n+2}[$, where $n > m$, is handled similarly.

b) ⇒ c) is proved by repeating the arguments in the proof of b) ⇒ c) in Theorem 3.2.2.

c) ⇒ d) By the assumptions,

$$E'_\beta = s \cdot \lim_{\to}(E/\operatorname{Ker} p_n, \widehat{p}_n)' = s \cdot \lim_{\to}(\operatorname{Ker} p_n^\perp, \|\cdot\|'_n)$$

and the representation $G'_\beta = s \cdot \lim_{\to}(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})'$ is valid for each nonnormable closed subspace $G \subset E$. But the space $(G/\operatorname{Ker} p_{n,G}, \widehat{p}_{n,G})$ is isometric to the subspace $k_n(G)$ of the quotient space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$, and this isometry is implemented by the correspondence $k_{n,G}(g) \to k_n(g)$, where $k_{n,G} : G \to G/\operatorname{Ker} p_{n,G}$ is the canonical mapping. Indeed, the equalities $\widehat{p}_{n,G}(k_{n,G}(g)) = p_{n,G}(g) = p_n(g) = \widehat{p}_n(k_{n,G})$ are valid and therefore $k_n(G)$ is closed subspace in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$. Further, $k_n(G)' = (E/\operatorname{Ker} p_n)'/k_n(G)^\perp$. But, as noted in the proof of statement c) of Theorems 2.3.1, $(E/\operatorname{Ker} p_n, \widehat{p}_n)' = \operatorname{Ker} p_n^\perp$, where $\operatorname{Ker} p_n^\perp$ is considered in the induced topology, which coincides with the topology $\|\cdot\|'_n$ with the unit ball V_n^0. Next, $k_n(G)^\perp = k_n^{'(-1)}(G^\perp) = \operatorname{Ker} p_n^\perp \cap G^\perp$ and $G'_\beta = s \cdot \lim_{\to} \operatorname{Ker} p_n^\perp/(\operatorname{Ker} p_n^\perp \cap G^\perp)$.

d) ⇒ e) From the conditions we immediately obtain that every closed nonnormable subspace G is a quojection. Indeed, it follows from statement a) of Theorem 2.3.2. This means that the quotient spaces $(G/\operatorname{Ker} p_{n,G},\ \widehat{p}_{n,G})$ are reflexive Banach spaces and therefore, by virtue of the above, $k_n(G)$ are closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ $(n \in \mathbb{N})$.

e) ⇒ a) Suppose that G is a closed nonnormable subspace of E, $x \in E \backslash G$ and $d(x, G) = r \in [\,2^{-n}, 2^{-n+1}[$. Then, by virtue of corollary of Proposition 3.1.4, the following equality is true:

$$\begin{aligned}\inf\{p_{n+1}(x-g); g \in G\} &= \begin{cases} 2^n r, & \text{when } \ r \in]\,2^{-n}, 2^{-n+1}[, \\ \lambda\ (\lambda \le 1), & \text{when } \ r = 2^{-n} \end{cases} \\ &= \inf\{\widehat{p}_{n+1}(k_{n+1}(x-g); g \in G\}.\end{aligned}$$

Since $k_{n+1}(G)$ is closed in the reflexive Banach space $(E/\operatorname{Ker} p_{n+1}, \widehat{p}_{n+1})$, by assumption, $k_{n+1}(G)$ is proximal with respect to the norm $\widehat{p}_{n+1}$. This implies that there exists an element $g_0 \in G$ such that

$$q_r(x - g_0) = \begin{cases} 1, & \text{when } \ r \in]\, 2^{-n}, 2^{-n+1}[, \\ \lambda\ (\lambda \leq 1), & \text{when } \ r = 2^{-n}. \end{cases}$$

Repeating now the arguments in the proof of c) ⇒ a) in Theorem 3.2.2, we get that G is proximal in E with respect to the metric (2.5.8). The proof of Theorem 3.4.1 is complete. □

Corollary 1. *Let a Fréchet space* $(E, \mathfrak{T})$ *satisfy one of the equivalent conditions of Theorem* 3.4.1*. Then each of its normable subspaces is also proximal in E with respect to the metric* (2.5.8)*.*

Indeed, let G be a closed normable subspace of $(E, \mathfrak{T})$ with respect to the seminorm p_{n_0}, i.e. the set $G \cap \{x \in E; p_{n_0}(x) \leq 1\}$ is a bounded neighborhood in G. Then $p_{n,G}$ is a norm on G and $(G/\operatorname{Ker} p_{n,G})$ is a Banach space isomorphic to the closed subspace $k_n(G)$ of the space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ for $n \geq n_0$. For the proximality of G, it suffices to prove that $k_n(G)$ is closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ also for $n < n_0$. To do this, we consider the closed subspace G_1 of E defined by $G_1 = k_{n_0}^{-1}\, k_{n_0}(G)$, where $k_{n_0} : E \to (E/\operatorname{Ker} p_{n_0}, \widehat{p}_{n_0})$ is the canonical mapping. The subspace G_1 is nonnormable, since it contains the subspace $\operatorname{Ker} p_{n_0}$ and $k_{n_0}(G_1) = k_{n_0}(G)$. Further, $k_n = \pi_{nm} \circ k_m$ $(m \geq n)$, where π_{nm} : $(E/\operatorname{Ker}_m, \widehat{p}_m) \to (E/\operatorname{Ker} p_n, \widehat{p}_n)$ is the canonical homomorphism. This implies that $k_n(G) = \pi_{nn_0} \circ k_{n_0}(G_1) = k_n(G_1)$ $(n \leq n_0)$, and the assertion is proved in view of the fact that $k_n(G_1)$ is closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$.

Corollary 2. *A Fréchet space* $(E, \mathfrak{T})$ *has the proximality property with respect to the metric* (2.5.8) *if and only if it is isomorphic to the space* $B \times \omega$*, where* B *is a reflexive Banach space.*

The validity of this consequence follows from the reasoning which were given before the formulation of Theorem 3.4.1 and Corollary 1.

Theorem 3.4.1 with its consequences is valid for normlike metrics d_1 and d_2. In particular, the proximality property has the Fréchet space $l^2 \times \omega$.

It should be noted that the space ω is the unique Fréchet space with the proximality property, which does not have infinite-dimensional Banach subspace. Indeed, it is well known that every infinite-dimensional subspace of ω is isomorphic to the space ω. In particular, this follows from Theorem 3.4.1, since every infinite-dimensional subspace of the space ω is nuclear and quojection.

Let now $(E, \mathfrak{T})$ be an arbitrary Fréchet space with the proximality property that does not have an infinite-dimensional Banach subspace, and G be its arbitrary infinite-dimensional subspace. By virtue of Theorem 3.4.1, G is a quojection and therefore on G there is no continuous norm. It follows from [16] that G contains a subspace isomorphic to the space ω. But every infinite-dimensional subspace G of ω contains a subspace isomorphic to the space ω only if E itself is isomorphic space ω [16].

By an insignificant modification of the above reasoning, it can be shown that Theorem 3.4.1 remains valid if in its equivalent statements the requirement “every closed nonnormed subspace” is replaced by the requirement “every closed subspace”.

It should also be noted that when proving the implication a) $\Rightarrow$ b) of Theorem 3.4.1, we asserted the closedness of the subspace H_G in G with respect to the restriction of the seminorm q_s to G. Since G is a nonnormed Fréchet space, we cannot assume that every its closed subspace is closed under some half-norms. More precisely, due to the results of [37], in any nonnormed Fréchet space, there is a closed subspace, not closed with respect to some seminorm.

Let us now prove that for the constructed by us metric we have a positive answer to the following question: does a metric exist on a metrizable space $E = \prod\limits_{k\in\mathbb{N}} (E_k, \|\cdot\|_k)$, where $(E_k, \|\cdot\|_k)$ are normed spaces with respect to which the product $G = \prod\limits_{k\in\mathbb{N}} G_k$ of proximal in $(E_k, \|\cdot\|_k)$ subspaces G_k is proximal.

Indeed, let us define the topology of the space E using the sequences of seminorms

$$p_n(x) = 2^{n-1}\sum_{k=1}^{n} \|x_k\|_k, \quad x = \{x_k\} \in E, \quad n \in \mathbb{N}.$$

From the proximality of G_k in E_k it follows that G is closed in E. Let $x = \{x_k\} \in E\backslash G$ and $d(x, G) = r \in [\,2^{-n}, 2^{-n+1}[$, then we have

$$\inf\{p_{n+1}(x-g); g \in G\} = 2^n \inf\left\{\sum_{k=1}^{n+1} \|x_k - g_k\|_k; g = \{g_k\} \in G\right\}$$

$$= 2^n \sum_{k=1}^{n+1} \|x_k - g_k^*\|_k = p_{n+1}(x-g^*) = \begin{cases} 2^n r, & \text{when } r \in]\,2^{-n}, 2^{-n+1}[\,, \\ \lambda\ (\le 1), & \text{when } r = 2^{-n}, \end{cases}$$

where $g^* = \{g_k^*\} \in G$ and g_k^* is the best approximation of x_k in G_k with respect to the norm $\|\cdot\|_k$ $(k = 1, \ldots, n+1)$, and $g_k^* \in G_k$ are arbitrary elements $(k = n+2, \ldots)$. Therefore, $d(x, g^*) = r$ and G is proximal in E with respect to the metric d.

3.4.1 Fréchet spaces with proximal hypersubspaces, but having non-proximal subspaces

From Theorem 3.4.1 it follows that the problem of finding a Fréchet space with proximal hypersubspaces, but having non-proximal subspaces, is equivalent to finding a reflexive quojection whose some subspace is not a quojection. In turn, this is equivalent to finding a reflexive strict (LB)-space, some quotient space of which is not strict (LB)-space, i.e. that part of question 3, posed in [43], which is solved negatively in [65]. That is why the mentioned counterexamples from [65] and [185] are identical and have the following form: let G be the nuclear Fréchet space, which is not isomorphic to the space ω. The space G is identified with the subspace of the product of Hilbert spaces $E = \prod\limits_{n\in\mathbb{N}} H_n$, and the space G'_β is identified with the quotient space $\oplus H_n/G^\perp$ of direct sum $\bigoplus_{n\in\mathbb{N}} H_n$ of sequence of Hilbert spaces that are not strict (LB)-spaces. Consequently, in the space E, every closed hypersubspace is proximal, but E has a nonproximal subspace. In particular, such is some hypersubspace of the space G. In connection with the task 3 from [43], it turns out that the reflexive strict (LB)-space has a quotient space that is not strict (LB)-space if and only if it is not isomorphic to the space $B \times \phi$, where B is a reflexive (B)-space, and $\phi = \omega'$ is the space of all finite sequences.

Product $(L^p(R))^N$ $(1 < p < \infty)$ of a sequence of reflexive Banach spaces, isomorphic to the space $L^p(R)$, is reflexive quojection. In the work [191], it was proven that the countably normed space $W^{p,\infty}(R)$ is isomorphic to the subspace $(L^p(R))^N$.

The given examples show that defined on these spaces any linear continuous functional $x' \in K_r^0$ attains on K_s $(s \le r)$ its supremum, but in these spaces there is a nonnormed closed subspace G such that for some $s \le r$ the restriction of some x' to G no longer attains on $K_s \cap G$ its supremum.

From Theorems 3.2.2 and 3.4.1 it follows that the following statements are true:

(i) In the strict Fréchet–Hilbert spaces E every closed hypersubspace is proximal with respect to normlike metrics (2.5.2) and (2.5.4) and metrics (2.5.8).

(ii) In Fréchet–Hilbert spaces E every closed subspace is proximal with respect to the mentioned metrics if and only if it is isomorphic to the space ω or $l^2 \times \omega$.

From these statements it turns out that in the space $(l^2)^N$ every closed hypersubspace is proximal with respect to the mentioned metrics. However, it contains nonproximal closed nonnormed subspaces. Let us now present a sufficient condition of proximality of closed subspaces of quojections.

Let E be a quojection with a sequence of seminorms $\{p_n\}$, where $2p_n \le p_{n+1}$ $(n \in \mathbb{N})$ and $p_1 \ne 0$. Consider E with the metric (2.5.8). An important property of this metric is that in the case of Fréchet–Hilbert spaces the seminorms q_r, which are Minkowski functionals for balls of this metrics K_r, are again generated by semiinner products, which, by virtue of (2.5.9), differ from the given ones only by positive factors. Let G be a closed subspace of the space E. Let us consider the quantity $\sup\{d(f, G);\ f \in E\}$. It is equal either to ∞, or 2^{-m+2} for some $m \geq 2$. This follows from the simple fact that if $d(f, g) = r \in I_n$ for some $f \in E$ and $g \in G$, then $\sup\{d(f, G);\ f \in E\} \geq \sup I_n$. From here we get the equality

$$\sup\{d(f, G); f \in E\} = r_m = \begin{cases} \infty, & \text{when } \ m = 1, \\ 2^{-m+2}, & \text{when } \ m \geq 2. \end{cases}$$

Proposition 3.4.2. *Let E be a reflexive quojection with the sequence of seminorms $\{p_n\}$, where $2p_n \leq p_{n+1}$ $(n \in \mathbb{N})$, $p_1 \neq 0$, and metric* (2.5.8). *Let also G be a closed subspace of E and $\sup\{d(f, G); f \in E\} = r_m$. If the set $k_n(G)$ is closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ for $n \geq m$, where $k_n : E \to E/\operatorname{Ker} p_n$ is a canonical mapping, then G is proximal in E with respect to the metric d.*

Proof. Let $f \in E \backslash G$ and $d(f, G) = r \in I_n$, then, by virtue of Proposition 3.1.4, the equality

$$\inf\{p_n(f - g); g \in G\} = \begin{cases} 2^{n-1} r, & \text{when } \ r \in \operatorname{int} I_n, \\ \lambda \leq 1, & \text{when } \ r = 2^{-n+1} \quad (n \in \mathbb{N}) \end{cases}$$

is true. Since $r < r_m$ and hence $n \geq m$, then $k_n(G)$ is closed in the reflexive Banach space $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ and is proximal in it with respect to the norm $\widehat{p}_n$. Now, repeating the reasoning, which was given in the proof of implication c) $\Rightarrow$ a) of Theorem 3.2.2, we find that G is proximal in E with respect to the metric d.

Similar reasoning can be used to prove this proposition for normlike metrics d_1 and d_2. □

Corollary. *Let E be a strict Fréchet–Hilbert space with the sequence of Hilbert seminorms $\{p_n\}$, where $2p_n \leq p_{n+1}$ $(n \in \mathbb{N})$, $p_1 \neq 0$, $(E/\operatorname{Ker} p_n, \widehat{p}_n) = H_n$, subspace $G = G_1 + \operatorname{Ker} p_s$, where $H_{m-1} \subset G_1 \subset H_m$, G_1 is a closed subspace of the spaces (H_m, p_{m,H_m}), $H_0 = \{0\}$ and $s \geq m$, $s \in \mathbb{N}$. Then G is proximal in E with respect to the metric* (2.5.8).

Proof. Consider the case when $G = G_1 + \operatorname{Ker} p_s$ $(s \geq 1)$, where G_1 is a closed subspace of the space H_1. Due to Corollaries of Theorem 2.4.3, G is a prequotient subspace of space E. Since the restriction of k_n to H_n is an isomorphism of H_n onto $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ for any n, then $k_n(G_1)$ is closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ for any $1 \leq n \leq s$. Let M_s be the topological complement of H_s to H_{s+1}, then $M_s \subset$

G and $G_1 + M_s$ is closed subspace of the space H_{s+1}. Obviously, $k_{s+1}(G) = k_{s+1}(G + M_s) = \pi^{-1}_{s,s+1}(k_s G_1)$. By the similar reasoning it will be proved that $k_n(G)$ is closed in $(E/\operatorname{Ker} p_n, \widehat{p}_n)$ for $n > s+1$. Further, since $\inf\{\widehat{p}_1(k_1 f_0 - k_1 g); g \in G_1\} = 1$ for some $f_0 \in H_1 \subset E$, we have $\sup\{d(f, G); f \in E\} = r_1 = \infty$. According to Proposition 3.4.2, G is proximal in E with respect to the metric d. The proof of the general case is immaterial differs from the considered one and we omit it. □

It is not known what the situation is in the case of non-adaptive information of the cardinality greater than one, since it is not known whether the arbitrary subspace of finite codimension of quojection is proximal. This issue has not even been solved in case of arbitrary strict Fréchet–Hilbert spaces. However, from the results of Section 3.4 it turns out that if $F_1 = X \times \omega$, where X is a reflexive Banach space, then for any neighborhood of zero F of the space F_1, the representation $F = B + \operatorname{Ker} \mu_F$ is valid, and for any non-adaptive information of any cardinality, there are interpolation splines. A similar statement is true for an arbitrary closed subspace of the space $X \times \omega$.

3.4.2 Proximality of finite-dimensional subspaces in metrizable locally convex spaces

Let (E, d) be a metric space, $G \subset E$, $f \in E \backslash G$ and $\{g_k\}$ be a sequence of elements from G such that

$$\lim_{k \to \infty} d(f, g_k) = d(f, G).$$

The sequence $\{g_k\}$ is said to be minimizing for f in G with respect to the metric d. Recall that G is called approximate compact in (E, d) if each minimizing sequence $\{g_k\}$ is compact in G, i.e. from $\{g_k\}$ we can extract a convergent subsequence. Approximate compactness and proximality of finite-dimensional subspaces in metrizable locally convex spaces with respect to the metrics (2.5.1), (2.5.2) and (2.5.4) have been discussed in [189, 191, 205]. Let us now present some results regarding the metric (2.5.8).

Proposition 3.4.3. *Let E be a metrizable LCS with a sequence of seminorms $\{p_n\}$, where $2p_n \le p_{n+1}$ $(n \in \mathbb{N})$ and $p_1 \not\equiv 0$. Then each of its finite-dimensional subspaces G is proximal in E with respect to the metric* (2.5.8).

Proof. Let $f \in E \backslash G$, $d(f, G) = r$, then as it was proved above, we have

$$\inf\{q_r(f - g); g \in G\} = \begin{cases} 1, & \text{when } r \in \bigcup\limits_{n \in \mathbb{N}} \operatorname{int} I_n, \\ \lambda, & \text{when } r = 2^{-n+1} \ (n \in \mathbb{N}), \end{cases}$$

where $0 \le \lambda \le 1$. If the restriction of q_r on G is a norm, then G is closed in E with respect to the norm q_r. Therefore, using the reasoning, which are well known for normed spaces, we obtain the existence of the best approximation g_0 with respect to q_r and, therefore, with respect to the metric (2.5.8).

If the restriction of q_r on G is not a norm, then $G \cap \operatorname{Ker} q_r = G_1 \neq \{0\}$. Let us decompose G as the sum of G_1 and the subspace M. Let $\{g_k\}$ be minimizing sequence for f in G with respect to q_r, i.e. $\lim\limits_{k\to\infty} q_r(f - g_k) = \lambda$. Let us represent each $g_k = g_k^{(1)} + m_k$, where $g_k^{(1)} \in G_1$ and $m_k \in M$. Then we have $\lim\limits_{k\to\infty} q_r(f - g_k) = \lim\limits_{k\to\infty} q_r(f - m_k)$. The sequence $\{m_k\}$ is bounded in G and therefore from it one can identify a subsequence converging to some element $m_0 \in G$. As is known, m_0 will then be the best approximation for f in G with respect to the metric d. Together with m_0, the best approximation for f in G with respect to q_r and with respect to the metric (2.5.8) is also each element of the form $m_0 + g$, where $g \in G_1$. Example 2 from Section 3.1 shows that λ can be zero for $r = 2^{-n+1}$ $(n \in \mathbb{N})$. In this case $f - m_0 \in \operatorname{Ker} q_r$. □

Proposition 3.4.4. *Let E be a metrizable LCS with a sequence of seminorms $\{p_n\}$, where $p_1 \not\equiv 0$, $2p_n \le p_{n+1}$ $(n \in \mathbb{N})$, and the metric* (2.5.8)*, G be its finite-dimensional subspace and* $\sup\{d(f, G);\ f \in E\} = r_m$. *$G$ is approximately compact in E with respect to the metric* (2.5.8) *if and only if the restriction of the seminorm p_m to G is a norm.*

Proof. Let $\{g_k\}$ be a minimizing sequence for f in G with respect to the metric d and

$$\lim_{k\to\infty} d(f, g_k) = d(f, G) = r \in I_n. \tag{3.4.1}$$

Obviously, $n \ge k$ and therefore the restriction of p_n to G is the norm. From the condition we obtain that for every $\varepsilon > 0$, there is $k_\varepsilon \in \mathbb{N}$ such that $d(f, g_k) < r + \varepsilon$ for $k \ge k_\varepsilon$, i.e. $f - g_k \in \operatorname{int} K_{r+\varepsilon}$. If $r + \varepsilon \in I_n$, then $f - g_k \in 2^{n-1} r(1 + \frac{\varepsilon}{r}) \operatorname{int} V_n$ and therefore $q_r(f - g_k) = 2^{-n+1} r^{-1} p_n(f - g_k) < 1 + \frac{\varepsilon}{r}$, when $k > k_\varepsilon$. So, the sequence $\{g_k\}$ is bounded by the seminorm q_r. Hence, due to the finite dimensionality of G and from the fact that the restrictions of q_r to G are norms, it follows the existence of a subsequence $\{g_{k_j}\}$ converging to some element $g_0 \in G$. Namely, g_0 is the best approximation for f in G.

Now, let G be approximately compact, $r \in \operatorname{int} I_n$ $(n \ge m)$ and for some $y \in G$ $(y \neq 0)$, the equality $q_r(y) = 0$ be valid. Then, by condition, for some $f \in E$, we have that $d(f, G) = r$ and therefore $\inf\{q_r(f - g; g \in G\} = 1$. Due to the finite-dimensionality of G, there exists $g_0 \in G$ such that $q_r(f - g_0) = 1$. Moreover, for arbitrary number λ, the following equality holds: $q_r(f - g_0 - \lambda y) = 1$

and therefore $d(f, g_0 + \lambda y) = r = d(f, G)$. If we now take the minimizing with respect to the metric sequence $g_k = g_0 + ky$, $k \in \mathbb{N}$, then it doesn't be compact.

It should be noted that for known normlike metrics, every minimizing with respect to the metric sequence is also minimizing for the corresponding Minkowski functional. By virtue of Proposition 3.1.4, this statement is true in the case of the metric (2.5.8) for the minimizing sequence $\{g_k\}$ satisfying (3.4.1) for $r \in \operatorname{int} I_n$. The example given after Proposition 3.1.4 shows that there is a sequence of approximately compact subspace, minimizing with respect to the metric that is not the same for the corresponding seminorm. $\square$

Note 3.4.1. In the work [5], the problem of the best approximation of a fixed function $f \in C(\mathbb{R})$ in the subspace G_m of polynomials whose degree does not exceed $m-1$, with respect to the metric (2.5.4), is studied. It reduces to the problem of uniformly best approximation the restriction of a function f on some segment $[a', b'] \subset \mathbb{R}$ by a subspace G_m with an upper semi-continuous weight function. At the same time, this weight function has a rather complex form and depends on the distance f to G_m.

We studied the same problem in the Fréchet space $C(]a, b[)$ of continuous functions on open interval $]a, b[$ with respect to the metric (3.1.4) and generalized the known results of the best approximation theory [78]. Due to the properties of this metric, it is proved that if $d(f, G_m) \neq 2^{-n+1}$ $(n \in \mathbb{N})$, this distance is equal to the best uniform approximation $\mathcal{E}_m(f; G_m; [a', b'])$ of the restriction f on a certain segment $[a', b'] \subset]a, b[$, without a weight function. Moreover, the polynomial of the best approximation with respect to the metric d is also the polynomial of the best uniform approximation on some $[a', b']$. It should be noted that a' and b' depend on f and m, and $\downarrow \lim_{m\to\infty} a' = a$, $\uparrow \lim_{m\to\infty} b' = b$.

In Section 3.1, Example 2 is given showing that if $f \in C(\mathbb{R})$, $d(f, G_2) = d(f, g_0) = 1$, then g_0 is not the best uniform approximation element on the segment $[-1, 1]$ with respect to the seminorm $|| \cdot ||_1$.

Some results obtained for the space $C(]a, b[)$ are extended to the space $C(T)$ of continuous functions, where T is a locally compact space countable at infinity.

3.5 On interpolation splines in the Fréchet space of differentiable locally integrable functions

In this section, we present sufficient conditions for the existence of interpolation splines in the Fréchet–Hilbert space of differentiable locally integrable functions $W^{2,k}_{loc}(R)$. To this end, we present the sufficient conditions for non-adaptive information I such that the finite defect subspace $\operatorname{Ker} I$ would be strongly proximal and the subspace $\operatorname{Ker} I$ has an orthogonal complement in the space $W^{2,k}_{loc}(R)$.

Theorem 3.5.1. *Let on the Fréchet–Hilbert space of k-differentiable $(k$ is fixed$))$ and in square locally integrable functions $F_1 = W^{2,k}_{loc}(R)$ an increasing $($with respect to $n)$ sequence of seminorms be given:*

$$\|f\|^{2,k}_n = \left(\sum_{s\le k}\int_{-n}^{n} |f^{(s)}(t)|\,dt\right)^{1/2}, \quad n \in \mathbb{N}, \tag{3.5.1}$$

$U_n = \{f \in W^{2,k}_{loc}(R);\ \|f\|^{2,k}_n \le 1\}$, $I : F_1 \to R^m$ is non-adaptive information $I(f) = [L_1(f), L_2(f), \dots, L_m(f)]$ of cardinality m, where $L_i \in F_1'$, $i = \overline{1,m}$. Then the following statements are valid:

a) *If $m = 1$, then for any $y \in R$ and arbitrary non-adaptive information in the space $W^{2,k}_{loc}(R)$, there is an interpolation spline.*

b) *If $k = 0$, then for arbitrary $y \in R^m$ and arbitrary non-adaptive information $I(f) = [L_1(f), L_2(f), \dots, L_m(f)]$, where the functionals L_i are generated by the functions g_i of the space $L^2[-1,1]$, in the space $W^{2,0}_{loc}(R) = L^2_{loc}(R)$ there is an interpolation spline.*

c) *If the closed subspace* $\operatorname{Ker} I$ *is a quojection, then for arbitrary $y \in R^m$ and arbitrary non-adaptive information in the space $W^{2,k}_{loc}(R)$, there exists an interpolation spline.*

Proof. a) Let

$$B = \Big\{f \in W^{2,k}_{loc};\ \|f\|^{2,k}_n = \left(\sum_{j\le k}\int_{-\infty}^{\infty} |f^{(j)}(t)|^2 dt\right)^{1/2} \le 1\Big\}.$$

It is easy to prove that $U^{p,k}_n \subset B + \operatorname{Ker}\|\cdot\|^{2,s}_n$ for any $n \ge 2$. Therefore, according to Theorem 2.3.1, the space $W^{2,k}_{loc}(R)$ is a reflexive quojection. For $m = 1$, the hyperspace $\operatorname{Ker} I$ is strongly proximal in the space $W^{2,k}_{loc}(R)$ (see Theorem 3.2.1) and therefore an interpolation spline exists (see Theorem 3.2.3).

b) For $k = 0$, we have $W^{2,0}_{loc}(R) = L^2_{loc}(R)$. Let $m = 2$ and $I(f) = [L_1(f), L_2(f)]$, where $L_i \in F_1'$, $i = 1,2$. It is known (see Theorem 2.4.1) that $L^2_{loc}(R) = L^2_0[-n,n] \dotplus \operatorname{Ker}\|\cdot\|^{2,0}_n$, where $L^2_0[-n,n]$ is the subspace of those functions from $L^2_{loc}(R)$ that are extended to the entire axis by zero. It is also known (see Theorem 2.4.1) that $F_1' = (L^2_{loc}(R))' = L^2_0(R) = \bigcup_{n=1}^{\infty} L^2_0[-n,n]$. In the case when the linear continuous functionals L_1 and L_2 are generated by the functions g_1 and g_2 from $L^2_0[-1,1]$, we will have that $\operatorname{Ker} I = \operatorname{Ker} L_1 \cap \operatorname{Ker} L_2$ will be a closed subspace of codimension 2 in $L^2_0[-1,1]$ and therefore a closed subspace in $L^2_{loc}(R)$.

Let us assume that $f \in L^2_{loc}(R)$. Consider the distance $d(f, \operatorname{Ker} I) = r$. Note that due to the properties of $\operatorname{Ker} I$, r can be an arbitrary positive number. When $r \in \operatorname{int} I_n$ $(n \in \mathbb{N})$, by Theorem 3.1.4, we also have that $\inf\{\|f - h\|_n^{2,0};\ h \in \operatorname{Ker} I\} = r \in \operatorname{int} I_n$. Since $\operatorname{Ker} I$ is a closed set in $L^2_0[-1, 1]$, it is also closed in $L^2_0[-n, n]$ for any n and therefore proximal, too. This means that for some $h_0 \in \operatorname{Ker} I$ we obtain $\|f - h_0\|_n^{2,0} = r$.

If $r = 2^{-n+1}$ $(n \in \mathbb{N})$, then, according to Proposition 3.1.4, we will have $\inf\{\|f - h\|_n^{2,0};\ h \in \operatorname{Ker} I\} = \lambda \le r = 2^{-n+1}$. If $0 < \lambda \le r$, there again exists $h_1 \in \operatorname{Ker} I$, which is the best approximation for f in $\operatorname{Ker} I$ with respect to the norm $\|\cdot\|_n^{2,0}$, i.e. $\|f - h_1\|_n^{2,0} = \lambda$. If $\lambda = 0$, we get $\inf\{\|f - h\|_n^{2,0};\ h \in \operatorname{Ker} I\} = 0$. This means that f is a continuation of some function from $\operatorname{Ker} I$ outside $[-n, n]$, i.e. $f - h_1 \in \operatorname{Ker} \|\cdot\|_n^{2,0}$ for some function h_1 from $\operatorname{Ker} I$.

Let us show that if $0 \le \lambda \le \|f - h_1\|_n^{2,0} < 2^{-n+1}$ for some function $h_1 \in \operatorname{Ker} I$, then h_1 is again the best approximation for f with respect to the metric d. Indeed, let us assume that $\|f - h_1\|_{n+1}^{2,0} < 2^{-n+1}$. Then, according to the definition of the metric (2.5.12), we also obtain that $d(f, h_1) \le \|f - h_1\|_{n+1}^{2,0} < 2^{-n+1}$, which is impossible, that means $\|f - h_1\|_{n+1}^{2,0} \ge 2^{-n+1}$. Therefore, again according to the definition of the metric (2.5.12), we obtain that $d(f, h_1) = 2^{-n+1} = r$ and therefore h_1 and h_2 are the best approximations for f in $\operatorname{Ker} I$ with respect to the metric d.

The general case when $I(f) = [L_1(f), \ldots, L_m(f)]$ is a non-adaptive information of cardinality m, where the functionals $L_i \in F_1'$ are generated by the functions from $L^2_0[-1, 1]$, does not significantly differ from the considered one and we omit it.

c) If the closed subspace $\operatorname{Ker} I$ is a quojection, then for arbitrary $y \in R^m$ and non-adaptive information in the space $W^{2,k}_{loc}(R)$, there exists a spline. To verify this, let us prove that the subspace $\operatorname{Ker} I$ is strongly proximal in $W^{2,k}_{loc}(R)$ with respect to the metric d. Let us assume that $f \in W^{2,k}_{loc}(R)$ and $d(f, \operatorname{Ker} I) = r \in I_n$. Then we also get

$$\inf\{\|f - h\|_n^{2,k};\ h \in \operatorname{Ker} I\} = \lambda \le r. \tag{3.5.2}$$

As noted above, if $r \in \operatorname{int} I_n$, then $\lambda = r$, and if $r = 2^{-n+1}$ $(n \in \mathbb{N})$, then $0 \le \lambda \le r = 2^{-n+1}$. Let $k_n : W^{2,k}_{loc}(R) \to W^{2,k}_{loc}(R)/\operatorname{Ker}\|\cdot\|_n^{2,k}$ be a canonical mapping and $\|\hat{\cdot}\|_n^{2,k}$ be an associated norm on the quotient space $W^{2,k}_{loc}(R)/\operatorname{Ker}\|\cdot\|_n^{2,k}$, which is defined according to the equality $\|k_n f\|_n^{2,k} = \|f\|_n^{2,k}$ for any function $f \in W^{2,k}_{loc}(R)$. In this notation (3.5.2) can be rewritten as follows:

$$\inf\{\|k_n f - k_n h\|_n^{2,k};\ h \in \operatorname{Ker} I\} = \lambda < r.$$

It is known that the subspace $k_n(\operatorname{Ker} I)$ is isometric to the normed space $(\operatorname{Ker} I/\operatorname{Ker} \|\cdot\|_{n,\operatorname{Ker} I}^{2,k})$, where $\|\cdot\|_{n,\operatorname{Ker} I}^{2,k}$ is the restriction of $\|\cdot\|_n^{2,k}$ on $\operatorname{Ker} I$. Since $\operatorname{Ker} I$ is a quojection, we obtain that $k_n(\operatorname{Ker} I)$ is a closed subspace in $W_{loc}^{2,k}(R)/\operatorname{Ker}\|\cdot\|_n^{2,k}$ in the topology $\|\widehat{\cdot}\|_n^{2,k}$ and therefore is also proximal. The rest of the proof is similar to case b). $\square$

Theorem 3.5.2. *Let us assume that* $F_1 = L_{loc}^2(R)$ *is considered with a sequence of seminorms* (3.5.1) *and with non-adaptive information* $I(f) = [L_1(f), \ldots, L_m(f)]$, *where* $L_i \in F_1'$, $i = 1, 2, \ldots, m$. *If the functionals* L_i *are generated by the functions* g_i *from the space* $L_0^2[-1, 1]$, *then the subspace* $\operatorname{Ker} I$ *has an orthogonal complement in* $L_{loc}^2(R)$.

Proof. Let $G_1 = \operatorname{Ker} I \cap L_0^2[-1, 1]$. By condition, G_1 is a closed subspace in the Hilbert space $L_0^2[-1, 1]$. Therefore, G_1 has an orthogonal complement $G_1^\perp$ in $L_0^2[-1, 1]$, which is finite-dimensional. It is also known [6] that the space $L_{loc}^2(R)$ can be represented as the topological sum of $L_0^2[-1, 1]$ and $\operatorname{Ker}\|\cdot\|_1^{2,0}$, i.e. $L_{loc}^2(R) = L_0^2[-1, 1] \dot{+} \operatorname{Ker}\|\cdot\|_1^{2,0}$. It is easy to prove that $\operatorname{Ker}\|\cdot\|_1^{2,0} \subset \operatorname{Ker} I$. Let us show that $L_{loc}^2(R)$ can be represented as an orthogonal sum of $\operatorname{Ker} I$ and $G_1^\perp$. Due to the above, we obtain $L_{loc}^2(R) = G_1 + G_1^\perp + \operatorname{Ker}\|\cdot\|_1^{2,0}$. We must prove that $G_1 + \operatorname{Ker}\|\cdot\|_1^{2,0} = \operatorname{Ker} I$. It is clear that $G_1 + \operatorname{Ker}\|\cdot\|_1^{2,0} \subset \operatorname{Ker} I$ and $G_1 + \operatorname{Ker}\|\cdot\|_1^{2,0}$ is a closed subspace in $L_{loc}^2(R)$.

Let $f \in \operatorname{Ker} I$. Then $f = g + z$, where $g \in L_0^2[-1, 1]$ and $z \in \operatorname{Ker}\|\cdot\|_{2,0,1}$. We also have that $g = g_1 + g_2$, where $g_1 \in G_1$ and $g_2 \in G_1^\perp$. From this we obtain $f = g_1 + g_2 + z = g_2 + y$, where $g_2 \in G_1^\perp$ and $y \in \operatorname{Ker} I$ $(y = g_1 + z)$. It remains to prove that $(g_1, y)_n^{2,0} = 0$ for all $n \in \mathbb{N}$. This follows from the fact that $(g_2, y)_n^{2,0} = (g_2, g_1 + z)_n^{2,0} = (g_2, g_1)_n^{2,0} + (g_2, z)_n^{2,0} = 0 + 0 = 0$. $\square$

An example of a one-dimensional subspace that has no orthogonal complement in $L_{loc}^2(R)$ is constructed in [84].

3.6 Definition of a central algorithm and condition for centrality of a spline algorithm in the Fréchet spaces

Let us introduce the definition of a central algorithm for the solution operator $S : F_1 \to G$, where F_1 is a linear space with a non-increasing sequence of absolutely convex subsets $\{V_n\}$ of the space F_1, and G is an LCS with the metric (3.1.4). Let I be non-adaptive information of cardinality $m \geq 1$, $y = I(F_1) = \mathbb{R}^m$, $I(f) = y$ $d(f, \operatorname{Ker} I) = r \in I_n$, that is, $f \in V_n \backslash V_{n+1}$ for some $n \in \mathbb{N}$. The quantity

$$e_n(\varphi, I, y) = \sup\{d(S(f), \varphi(y)); f \in I^{-1}(y) \cap V_n\}$$

we call the local error of the algorithm φ at point y. Let $r_n(I,y)$ denote the local radius of non-adaptive information I at point y, which is defined by the equality

$$r_n(I,y) = \text{rad}\ (S(I^{-1}(y) \cap V_n)).$$

Here the radius of the set $M \subset G$ is defined similarly to the case of a normed space according to the equality $\text{rad}(M) = \inf\{\sup\{d(a,g);\ a \in M\};\ g \in G\}$. The Chebyshev center $c \in G$ of the set $M \subset G$ is defined by the equality $\text{rad}(M) = \sup\{d(a,c),\ a \in M\}$. It is easy to see that $r_n(I,y) = \inf\{e_n(\varphi, I, y) : \varphi \in \Phi\}$, where Φ is the set of all algorithms. Global radius $r_n(I)$ of non-adaptive information I is defined by

$$r_n(I) = \sup\{r_n(I,y);\ y \in I(V_n)\}.$$

Let $y \in I(F) \subset \mathbb{R}^m$, that is, $y \in I(V_n)$ for some $n \in \mathbb{N}$. Let us assume that the sets $S(I^{-1}(y) \cap V_k)$ have the Chebyshev center $c = c(y)$ for all $y \in I(F)$ and $k \leq n$ if $y \in I(V_n)$. This means that for all $k \leq n$,

$$\begin{aligned} \text{rad}(S(I^{-1}(y) \cap V_k) &= \inf\{\sup\{|S(f) - g|;\ f \in I^{-1}(y \cap V_k)\};\ g \in G\}) \\ &= \sup\{|S(f) - c(y)|;\ f \in I^{-1}(y) \cap V_k\}. \end{aligned}$$

In this case, we call the algorithm $\varphi^c(y) = c(y)$, $y \in I(F)$, central. If G is a normed space,$V_1 = V_2 = \cdots = F$, $|\cdot| = q_F(\cdot)$, then this definition of centrality coincides with the classical definition.

The algorithm is called optimal error algorithm if

$$e_n(\varphi^*, I) = \inf\{e_n(\varphi, I);\ \varphi \in \Phi\}\,,$$

where Φ is the set of all algorithm and $n \in \mathbb{N}$.

The n-th global error of the algorithm φ will be called

$$e_n(\varphi, I) = \sup\{e(\varphi, I, y);\ y \in I(V_n)\}\,.$$

Due to the remark made in ([158], p. 49) regarding to the optimal algorithm φ^*, we have the inequalities

$$e_n(\varphi^*, I, y) \geq r_n(I,y) \text{ for each } y \in R^m \text{ and } e_n(\varphi^*, I) \geq r_n(I)\,.$$

Similar to the classical case for the central algorithm, in the Fréchet space the following equalities are also valid:

$$e_n(\varphi^c, I, y) = r_n(I,y) \quad \text{and} \quad e_n(\varphi^c, I) = r_n(I)$$

Consider a metrizable LCS whose topology is determined by some non-decreasing sequence of seminorms $\{\|\cdot\|_n\}$. Later, by d^* we will denote one of the following metrics: 1) metric defined by (3.1.4); 2) normlike metric (2.5.2) constructed by Albinus; 3) supremum metric defined by the formula $d(x,y) = \sup_{n\in\mathbb{N}} \frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)}$; 4) metric (2.5.1) constructed by Mazur.

Proposition 3.6.1. *Let G be a metrizable LCS with the metric d^* and let the closure $\bar{A}$ of $A \subset G$ be symmetric with respect to some element $p \in G$. Then p is the Chebyshev center for A.*

Proof. Consider the case where d^* is the metric defined according to (3.1.4) and assume that p is not the Chebyshev center of the set A. Then there is an element u of G such that $\sup\{|a-u| : a \in A\} < \sup\{|a-p| : a \in A\}$, where $|\cdot|$ is a quasinorm of the metrics (3.1.4). Let us select $x \in A$ such that

$$|a-u| < |x-p| \quad \text{for all} \quad a \in \bar{A}\,. \tag{3.6.1}$$

Let $|x-p| = r \in I_n$ for some $n \in \mathbb{N}$. If $r \in \operatorname{int} I_n$, then $|x-p| = \|x-p\|_n$. Let $|a_0-u| = r_1$ for some $a_0 \in A$ and $r_1 \in I_{n_1},\ n_1 \geq n$. If $r_1 \in \operatorname{int} I_{n_1}$, then $r_1 = \|a_0-u\|_{n_1}$. In this case, according to (3.6.1), we obtain $\|a_0-u\|_{n_1} < \|x-p\|_n$ and $\|a_0-u\|_n \leq \|a_0-u\|_{n_1} < \|x-p\|_n$. So,

$$\|a_0-u\|_n < \|x-p\|_n\,. \tag{3.6.2}$$

Let now $|a_0-u| = r_1 = 2^{-n_1+1} \in I_{n_1},\ n_1 \geq n$. From the properties of the metric (3.1.4) it follows that $\|a_0-u\|_{n_1} \leq 2^{-n_1+1}$. Therefore, $\|a_0-u\|_n \leq \|a_0-u\|_{n_1} \leq r_1 = 2^{-n_1+1} < r = \|x-p\|_n$ and the inequality (3.6.2) is true.

Let us now consider the case when $|x-p| = r = 2^{-n+1} \in I_n$ and $r_1 \in \operatorname{int} I_{n_1},\ n_1 > n$. Then $|a_0-u| = r_1 = \|a_0-u\|_{n_1} \geq \|a_0-u\|_{n+1}$ and therefore $\|a_0-u\|_{n+1} < 2^{-n+1}$. From the properties of the metric (3.1.4) it follows that $\|x-p\|_n \leq 2^{-n+1} \leq \|x-p\|_{n+1}$. It turns out that $\|a_0-u\|_{n+1} < 2^{-n+1} \leq \|x-p\|_{n+1}$ and (3.6.2) is valid for $n+1$.

It remains to consider the case when $r = 2^{-n+1} \in I_n$ and $r_1 = 2^{-n_1},\ n_1 \geq n$. Then we have $\|a_0-u\|_{n+1} \leq \|a_0-u\|_{n_1+1} \leq r_1 = 2^{-n_1} < r = 2^{-n+1} \leq \|x-p\|_{n+1}$ and hence (3.6.2) is also valid in the case $n+1$.

Now let d^* be a normlike, supremum, or Mazur-constructed metric. Then (3.1.4) is valid for some $n \in \mathbb{N}$. Indeed, if we assume that (3.6.2) is not true, then $\|a-u\|_n \geq \|x-p\|_n$ for all n and $a \in A$. From the properties of the metrics under consideration it follows that $|a-u| \geq |x-p|$ for all $a \in A$, but this contradicts the inequality (3.6.1). Let $x = p+h$. For $x \in A$ and $\bar{A}$ is symmetric with respect to p, $\bar{A}$ also includes $p-h$. If $k = n$ or $k = n+1$, we have

$$2\|h\|_k = \|2h\|_k = \|(p+h-u)-(p-h-u)\|_k$$

$$\leq \|(p+h)-u\|_k + \|(p-h)-u\|_k < 2\|x-p\|_k = 2\|h\|_k\,,$$

and it turns out to be a contradiction. □

Suppose that the topology of the Fréchet space E is given by a sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, that is, each seminorm $\|\cdot\|_n$ is generated by the semiinner product $(x,y)_n$ and $V_n = \{x \in E;\ \|x\|_n \leq 1\}$. In such spaces the concept of orthogonality is naturally introduced in Section 2.4: elements $x, y \in E$ are called orthogonal if $(x,y)_n = 0$ for all $n \in \mathbb{N}$. A subspace M has an orthogonal complement $M^{\perp}$ in E if each element $x \in E$ can be represented as a sum $x = y+z$, where $y \in M,\ z \in M^{\perp}$ and $(y,z)_n = 0$ for each $n \in \mathbb{N}$. In other words, this means that in the subspaces M and $M^{\perp}$ each element $x \in E$ has a unique element of the best approximation y and z, respectively, with respect to all seminorms $\|\cdot\|_n$, generated by $(\cdot,\cdot)_n$.

Theorem 3.6.2. *Let E be a Fréchet space with a non-decreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, $V_n = \{x \in E : \|x\|_n \leq 1\}$ and with metric* (3.1.4)*. Let $K_n : E \to E/\operatorname{Ker}\|\cdot\|_n$ be the canonical mapping $X_n = (E/\operatorname{Ker}\|\cdot\|_n\ ,\ \widehat{\|\cdot\|_n})$ and G be a metrizable LCS, $S : E \to G$ be a linear operator, and I be non-adaptive information of cardinality $m \geq 1$. Then the following statements are valid:*

a) *If $K_n(\operatorname{Ker} I)$ is closed in the Hilbert space X_n, $n \in \mathbb{N}$, then* $\operatorname{Ker} I$ *is strongly proximal in E with respect to the metric* (3.1.4)*, and for any $y \in I(E)$ there is a spline σ interpolatory y.*

b) *If, moreover, the subspace* $\operatorname{Ker} I$ *has an orthogonal complement in E, then for any $y \in I(E)$ there exists the unique spline σ interpolatory y such that $(\sigma,h)_n = 0$ for any $n \in \mathbb{N}$ and $h \in \operatorname{Ker} I$. If $y \in I(V_1)$, then σ is a center of all sets $I^{-1} \cap V_k$ for which these intersections are nonempty. The corresponding spline algorithm $\varphi^s(y) = S(\sigma)$ is linear and central.*

Proof. a) For $y \in I(E)$, there exists $f \in E$ such that $I(f) = y$. The subspace $\operatorname{Ker} I$ is strongly proximal in E and for this f there exists a strongly best approximation element h^* in $\operatorname{Ker} I$. Then $\sigma = f - h^*$ is a spline interpolatory y.

b) If $y = 0$, then likewise $\sigma = 0$ and item b) is trivial. For any nontrivial $y \in I(E)$ and information I, we take f such that $I(f) = y$. Since the subspace $\operatorname{Ker} I$ possesses an orthogonal complement in E, there exists the unique representation $f = h^* + \sigma$ and $(h^*,\sigma)_n = 0$ for any $n \in \mathbb{N}$, where $h^* \in \operatorname{Ker} I$ and $\sigma \in \operatorname{Ker} I^{\perp}$. This means that $\langle K_n h^*, K_n \sigma\rangle_n = 0$ for any $n \in \mathbb{N}$, where $\langle\cdot,\cdot\rangle_n$ is the inner product in the space X_n, generating the associative norm $\widehat{\|\cdot\|}_n$. $K_n(\sigma)$ is orthogonal to $K_n(\operatorname{Ker} I)$ in X_n for any $n \in \mathbb{N}$ and σ is a best approximation element for f in $\operatorname{Ker} I^{\perp}$ with respect to the $\|\cdot\|_n$ for any $n \in \mathbb{N}$. It is clear that

$I(\sigma) = y$. Let us prove that σ is a spline interpolatory y. Let $d(f, \operatorname{Ker} I) = r \in I_n$ for some $n \in \mathbb{N}$, then

$$\inf\{\|K_n f - K_n h\|_n \,,\; h \in \operatorname{Ker} I\} = \|K_n f - K_n h^*\|_n = \|\sigma\|_n = \lambda.$$

If $r \in \operatorname{int} I_n$, then, according to Proposition 3.6.1, $\lambda = r = d(f, h^*)$ and σ is a spline interpolatory y. If $r = 2^{-n+1}$ $(n \in \mathbb{N})$, then again by Proposition 3.6.1, $d(f, h^*) = r$ and σ is a spline interpolatory y. If $r = 2^{-n+1}$ and $\lambda = 0$, then $f - h_0 \in V_n$ for some $h_0 \in \operatorname{Ker} I$. Indeed, in this case there exists a minimized sequence $\{h_k\}$ such that $\lim_{k\to\infty} \|K_n f - K_n h_k\|_n = \lambda = 0$. Since $K_n(\operatorname{Ker} I)$ is closed in X_n, we have $K_n f \in K_n(\operatorname{Ker} I)$, i.e., $f \in \operatorname{Ker} I$. But this is out of the question and hence $\lambda = 0$ is impossible.

We obtain that $f - h_0 \in V_n$. Assuming now that $f - h_0 \in 2 \operatorname{int} V_{n+1}$, we will have $d(f, h_0) \le 2 \cdot 2^{-n} \|f - h_0\|_{n+1} < 2^{-n+1} = r$, but this is impossible. Therefore, $\|f - h_0\|_n \le 2^{-n+1}$ and $\|f - h_0\|_{n+1} \ge 2^{-n+1}$. This implies that $f - h_0 = r = 2^{-n+1}$.

From the above-said it follows that if some element $\sigma \in E$ satisfies the equalities $I(\sigma) = y$ and $(\sigma, h)_n = 0$ for any $n \in \mathbb{N}$ and $h \in \operatorname{Ker} I$, then σ is a spline interpolatory y.

Build now a linear spline algorithm. Towards this,we apply the method considered in ([158], p. 79). Let σ_i be the unique spline interpolatory $e_i = \{0, \dots, 1, \dots, 0\}$ for the information $I(f) = [L_1(f), \dots, L_m(f)]$ with linearly independent linear functionals $L_i(f)$ such that $K_n(\sigma_i)$ is orthogonal to $K_n(\operatorname{Ker} I)$ in X_n for any $n \in \mathbb{N}$, i.e., $(h, \sigma_i)_n = 0$ for all $n \in \mathbb{N}$. Consider the expression $\sigma = \sum_{i=1}^m L_i(f)\sigma_i$. Then $K_n(\sigma)$ will be orthogonal to $K_n(\operatorname{Ker} I)$ in X_n for all $n \in \mathbb{N}$ and σ will be a spline interpolatory y. It is clear that $\varphi^s(I(f)) = \sum_{i=1}^m L_i(f) S\sigma_i$ will be a linear algorithm. It should also be noted that the operator $y \to \sigma$, acting from the finite dimensional space $I(E)$ to the finite dimensional space $(\operatorname{Ker} I)^\perp$, is linear. It remains to prove that φ^s is central, i.e., the center of the set $S(I^{-1}(y) \cap V_{n_0})$ for each $y \in I(V_{n_0})$ is $S(\sigma)$, where σ is the above-mentioned unique spline interpolatory y. The existence of such spline σ was proven above. We have now to prove that if g is an arbitrary element of $I^{-1}(y) \cap V_{n_0}$, then $2\sigma - g \in I^{-1}(y) \cap V_{n_0}$. This fact may be proved just in the same way as in ([158], p. 97). Really, for $h = \sigma - g \in \operatorname{Ker} I$ we have

$$\begin{aligned}\|K_{n_0}\widehat{(2\sigma - g)}\|_{n_0} &= \|K_{n_0}\widehat{(\sigma + g)}\|_{n_0} = \sqrt{\|\widehat{K_{n_0}(\sigma)}\|_{n_0}^2 + \|\widehat{K_{n_0}(h)}\|_{n_0}^2} \\ &= \|\widehat{K_{n_0}(h)}\|_{n_0} = \|g\|_{n_0} \le 1\,,\end{aligned}$$

that is, $2\sigma - g \in I^{-1}(y) \cap V_{n_0}$. Therefore, the set $S(I^{-1}(y) \cap V_{n_0})$ is symmetric with respect to $\varphi^s(y) = S(\sigma)$, i.e., $\operatorname{rad}(S(I^{-1}(y) \cap V_k)) = \inf\{\sup\{|S(f) -$

$y|;\ f \in I^{-1}(y) \cap V_k\};\ y \in G\} = \sup\{|S(f) - S(\sigma)|;\ f \in I^{-1}(y) \cap V_k\}$, for all $k \leq n_0$. From the above and from Proposition 3.6.1 it follows that the spline σ interpolatory y is the center of the set $I^{-1}(y) \cap V_{n_0}$, i.e., $\mathrm{rad}(I^{-1}(y) \cap V_k) = \inf\{\sup\{|f - q|;\ f \in I^{-1}(y) \cap V_k\}; q \in E\} = \sup\{|f - q|;\ f \in I^{-1}(y) \cap V_k\}$ for all $k \leq n_0$. $\square$

CHAPTER 4

Central spline algorithms of projection (least squares and Ritz) methods in Fréchet–Hilbert spaces and its applications

4.1 Linear equations in Fréchet spaces

Consider an equation

$$Au = f\,, \tag{4.1.1}$$

where $A : D(A) \subset G \to E$ is a linear operator that maps an everywhere dense subset $D(A)$ of a Fréchet space G into a Fréchet space E. The main results have been obtained in the case when the topology E is given by a non-decreasing sequence of hilbertian norms $\{\|\cdot\|_n\}$, i.e. $\|x\|_n = (x,x)_n^{1/2}$ for each $x \in E$, where $(\cdot,\cdot)_n$ is the inner product on E for each $n \in \mathbb{N}$. Such spaces are, for example, the complete countable-Hilbert spaces or the nuclear Fréchet spaces with continuous norms that are well known in the theory of generalized functions from Section 2.6 (an example of a nuclear space with continuous norm that is not countable-Hilbert is constructed in [51]).

Linear equations in Fréchet spaces (and also in more general locally convex spaces) have been carefully studied in many monographs (see, for example, [83, 134]). However, approximate methods for the solution of linear equations in these spaces, as far as we know, have not been studied up to now. In [134], the approximation methods in countable-normed spaces are considered, based on the continuity of embeddings of such spaces in Banach spaces. But there are Fréchet spaces that do not have this property.

In this chapter, we use the metric (2.5.8) to generalize the classical least squares method [99] for the approximate solution of the equation (4.1.1) in the Fréchet space. Namely, in the classical case of Hilbert spaces, approximate solutions are found by minimizing the discrepancy with respect to the inner product on some finite-dimensional subspace. In the case of Fréchet space, approximate solutions are found by minimizing the discrepancy on the mentioned subspace with respect to this metric, which in the case of Hilbert spaces coincides with the metric generated by the inner product. Therefore, due to the above-mentioned property of the metric (2.5.8), the approximate solution again satisfies the system of linear algebraic equations.

This chapter examines the symmetric and self-adjoint operators in Fréchet–Hilbert spaces. For such operators, the well-known Theorems of von Neumann (Theorem 4.2.2), Hellinger–Toeplitz (part a) of Theorem 4.2.2) and Friedrichs, Stone, Wintner (Theorem 4.2.3) are generalized. Fréchet–Hilbert spaces provide natural extensions of symmetric and self-adjoint operators, which are important in quantum mechanics and mathematical physics. In particular, the position operator, momentum operator, harmonic oscillator operator and others continue from the space $L_2(\mathbb{R})$ into strict Fréchet–Hilbert spaces in various ways. Continued operators, in many cases, turn out to be continuous. The well-known Ritz method (Theorem 4.3.4) is generalized for operator equations in Fréchet–Hilbert spaces and some estimates are given. A natural definition of the operator A^∞ is given, which is essentially used in what follows. According to this generalization, $D(A^\infty)$ takes on a new meaning. The centrality of the Ritz method in the Fréchet space $D(A^\infty)$ (Theorem 4.4.5) is proved. This result is used for the approximate solution of strongly degenerate elliptic operators, for the Sturm–Liouville problem, the Laplace-Beltrami operator, etc. The extended Ritz method is used to approximate solution of the equation (4.4.7) in the space $D(A^\infty)$. The eigenfunctions of the operator A are chosen as basis functions and it is proved that the subspaces spanned by the first m eigenvectors have an orthogonal complement in the Fréchet space $D(A^\infty)$. This means that approximate solutions do not depend on the number of norms generating the topology of the space $D(A^\infty)$. The convergence of a sequence of approximate solutions to the exact solution is proved in the space $D(A^\infty)$, the topology of which is stronger than the topology of the original Hilbert space (Theorem 4.4.7). It is proved that the algorithm given in Theorem 4.4.5 is both a central and a spline algorithm. For the harmonic oscillator operator, the space $D(A^\infty)$ coincides with the Schwartz space $S(\mathbb{R})$ of rapidly decreasing functions. Therefore, if we take the eigenfunctions of this operator as basis functions, i.e. the Hermite functions, then the sequence of approximate solutions converges to the exact solution in the space $S(\mathbb{R})$, the topology of which is stronger than the topology of the Sobolev space.

4.1.1 Method of least squares for operator equation in the Fréchet–Hilbert spaces

Let E and G be Fréchet spaces with fixed non-decreasing sequences of seminorms $\{\|\cdot\|_n\}$ and $\{|\cdot|\}_n$, respectively. A linear operator $T : E \to G$ is called continuous if for any seminorm $|\cdot|_n$ there is a seminorm $\|\cdot\|_k$ and a number $c_n > 0$ such that

$$|Tx|_n \leq c_n \|x\|_k \quad \text{for every} \;\; x \in E.$$

The space of all linear continuous operators from E to G is denoted by $L(E, G)$. By analogy with [**?**], for the operator $T \in L(E, G)$, we introduce the function $\sigma_{_T} : N \to N$, which characterizes the continuity of T. The function $\sigma_{_T}$ is defined using the equality

$$\sigma_{_T}(n) = \inf\left\{\sigma \in \mathbb{N};\; \sup\left\{|Tx|_n;\; \|x\|_\sigma \leq 1\right\} < \infty\right\}. \tag{4.1.2}$$

According to ([11], see also [129]), a linear operator is called *tame* if there exist $\ell \in \mathbb{N}$ and a constant $c_n > 0$ such that

$$|Tx|_n \leq c_n \|x\|_{n+\ell} \quad \text{for every} \;\; x \in E, \tag{4.1.3}$$

i.e. in this case $\sigma_{_T}(n) \leq n + \ell$ for some $\ell \in \mathbb{N}$. The operator T is called *isometrically tame* if $\ell = 0$. The set of all such operators is denoted by $L_0(E, G)$. The tame operators are defined similarly in the case of incomplete metrizable locally convex spaces E and G. The invertibility of the operator $A : G \to E$ means that for any seminorm $|\cdot|_n$ on G there exist a seminorm $\|\cdot\|_{\sigma'(n)}$ and $C_n > 0$ such that

$$|g|_n \leq C_n \|Ag\|_{\sigma'(n)}\,, \tag{4.1.4}$$

where $\sigma'(n)$ is a function characterizing the continuity of the inverse operator A^{-1}.

A linear operator $A : G \to E$ is said to be *tame invertible* if it has an inverse tame operator. The tame invertibility of the operator A means that for any seminorm $|\cdot|_n$ on G there exist $\ell \in \mathbb{N}$ and a constant $C_n > 0$ such that

$$|g|_n \leq C_n \|Ag\|_{n+\ell} \quad \text{for every} \;\; g \in G.$$

A linear operator T is called a *tame isomorphism* if T is bijective and T and T^{-1} are tame. Two sequences of seminorms generating the topology of the space E are said to be *tame equivalent* if the identity map is a tame isomorphism. Examples of tamely equivalent and nonequivalent sequences of seminorms are given in [129]. You can also find examples of tame isomorphisms there.

There are examples of Fréchet spaces E and operators $T \in L(E, G)$ for which $\sigma_{_T}(n)$, generally speaking, differs from n. The set of all operators T from

$L_0(E, G)$ for which c_n does not depend on n is denoted by $L_F(E, G)$ [25]. For $E = G$, these operator classes were defined in [84] and denoted by $L_0(E)$ and $L_F(E)$, respectively. As was proven in [85], $L_0(E)$ is a complete m-convex algebra, and $L_F(E)$ is a Banach algebra.

A Fréchet space is called tame if there exists an increasing function $S : N \to N$ such that for every continuous operator T, the inequality $\sigma_T(n) \leq S(n)$ holds for all n starting from some, where $\sigma_T(n)$ are defined by equality (4.1.2). This is equivalent to the fact that there exist the increasing functions S_k $(k \in \mathbb{N})$ such that for each operator T there exists k_0 such that $\sigma_T(n) \leq S_{k_0}(n)$. Obviously, this definition does not depend on the choice of the sequence of seminorms generating the topology of the Fréchet space.

There are examples of the Fréchet space E and operators $T \in \mathcal{L}(E, F)$ for which $\sigma_T(n)$, generally speaking, differs from n. The set of all operators T from $\mathcal{L}(E, G)$ for which $\sigma_T(n) = n$ is denoted by $\mathcal{L}_0(E, G)$. And the set of operators for which C_n does not depend on n is denoted by $\mathcal{L}_F(E, G)$. For $E = G$, these classes of operators were defined in ([100], p. 59) and were denoted by $\mathcal{L}(E)$, $\mathcal{L}_0(E)$ and $\mathcal{L}_F(E)$, respectively.

4.1.2 Definition of approximate solution and convergence of its sequence

We study the equation (4.1.1) with linear operators $A : G \to E$, for which $A^{-1} \in \mathcal{L}(E, G)$ and (4.1.4) is fulfilled.

By analogy with the definition of an A-complete sequence in Banach spaces [99], we introduce the following definition: a sequence of the above-mentioned basis functions $\{g_i\}$ from G is called A-complete in a Fréchet space E if for any $\varepsilon > 0$ and $g \in D(A)$ there exist $n_0 = n_0(g, \varepsilon)$ and $\alpha_1, \ldots, \alpha_{n_0}$ such that

$$\left| Ag - \sum_{i=1}^{n_0} \alpha_i A g_i \right| < \varepsilon ,$$

i.e. $A(G) \subset \overline{\cup_{m=1}^{\infty} A(G_m)}$, where $\overline{M}^E$ means the closure of the set M in E, where $|\cdot|$ is the quasinorm of the metric (3.1.4) on G.

In particular, if $\{g_i\}$ is a basis of the Fréchet space G belonging to the set $D(A)$, then the sequence $\{G_m\}$ is limit dense in G.

Let us denote by J the discrepancy for the equation (4.1.1), i.e. the functional defined by the equality

$$J(g) = d(Ag, f) = |Ag - f| .$$

We will also denote by $J_n(g)$ the discrepancy with respect to the norm $\|Ag - f\|_n$.

An approximate solution of the equation (4.1.1) will be called

$$u_m = \sum_{i=1}^{m} \alpha_i g_i \in G_m$$

if it provides a minimum of the discrepancy $J_n(g)$, where n is chosen from the relation

$$\inf\{J(g);\ g \in G_m\} \in I_n.$$

Lemma 4.1.1. *If in the notation introduced above the equality* $\inf\{J_n(g);\ g \in M\} = r$ *holds for some subset* $M \subset G$ *and* $r \in I_n$, *then* $\inf\{J(g);\ g \in M\} = r$. *If also* $r \in I_n$, *and* $r \neq 2^{-n+1}$ $(n \in \mathbb{N})$, *then the converse is also true.*

There are examples which show that if $\inf\{J(g);\ g \in M\} = 2^{-n+1}$ $(n \in \mathbb{N})$, then $\inf\{J_n(g);\ g \in M\} = s < 2^{-n+1}$. Such examples in the case of the identity operator $A = I$, i.e. for the best approximation problem, were given in Section 3.1.

Corollary. *The following statements hold:*

a) *If* $\inf\{J(g);\ g \in M\} = r \in \operatorname{int} I_n$ $(n \in \mathbb{N})$, *then*

$$\inf\{J_n(g);\ g \in M\} = \inf\{J(g);\ g \in M\}\,. \tag{4.1.5}$$

b *If* $\inf\{J(g);\ g \in M\} = 2^{-n+1}$ $(n \in \mathbb{N})$, *then*

$$\inf\{J_n(g);\ g \in M\} \leq \inf\{J(g);\ g \in M\}\,.$$

Lemma 4.1.2. *Suppose the topology of a Frecher space* E *is induced by a non-decreasing sequence of hilbertian norms* $\{\|\cdot\|_n\}$ *and the homogeneous equation* $Ag = 0$ *has a unique solution, that is,* A *is injective. Then an approximate solution* $u_m \in G_m$ *for* (4.1.1) *can be constructed for each* $m \in \mathbb{N}$ *using the equation*

$$\inf\{J_n(g);\ g \in G_m\} = J_n(u_m) \tag{4.1.6}$$

provided that

$$\inf\{J(g);\ g \in M\} \in I_n,$$

and it is defined in a unique manner.

Proof. Let $m \in \mathbb{N}$ and $\inf\{J(g);\ g \in G_m\} = r \in I_n$. If $r \in \operatorname{int} I_n$, then, using Lemma 4.1.1, we also have that for some $u_m \in G_m$,

$$\inf\{J_n(g);\ g \in G_m\} = J_n(u_m)\,.$$

The norm $\|\cdot\|_n$ is generated by the inner product $(\cdot,\cdot)_n$. We remark that the positive function $J_n(g)$ attains its minimum on G_m at some point u_m if and only if its square $J_n^2(g)$ attains its minimum at this point. But as is well known ([99], p. 57), $J_n^2(g)$ attains its minimum at the function $u_m = \sum_{i=1}^m a_i^{(m)} g_i$, with coefficients that satisfy the system

$$\sum_{i=1}^{m} a_i^{(m)} (Ag_i, Ag_k)_n = (f, Ag_k)_n,\ k = 1, \ldots, m\,. \tag{4.1.7}$$

Since A is injective, it follows that the functions $Ag_1, \ldots, Ag_m$ are linearly independent for any $m \in \mathbb{N}$. It is well known that the necessary and sufficient condition for the linear independence of the system $\{Ag_i\}_{i=1}^m$ is that the Gram determinant does not vanish: $\det(Ag_i, Ag_k)_n = G(Ag_1, \ldots, Ag_m)_n \neq 0$, where $n = n(m)$. Hence, the determinant of the system is not zero, (4.1.7) has a unique solution for any $m \in \mathbb{N}$. If now $r = 2^{-n+1}$ $(n \in \mathbb{N})$, then we have also found the solution of the system (4.1.7) and the solution $u_m \in G_m$, satisfies (4.1.6). Moreover, by Lemma 4.1.1, $J_n(u_m) \leq r$. If $J_n(u_m) = r$, then again applying Lemma 4.1.2 we have that $J(u_m) \leq r$. So, suppose that $J_n(u_m) < r$, that is, $\|Au_m - f\|_n < r = 2^{-n+1}$. This implies that $Au_m - f \in \operatorname{int} K_r = \operatorname{int} V_n$. We assume that $Au_m - f \in 2 \operatorname{int} V_{n+1}$. Then we find that

$$J(u_m) \leq 2^{-n} p_{n+1}(Au_m - f) < 2^{-n} \cdot 2 = 2^{-n+1} = r,$$

which is not possible. This means that $Au_m - f \in \operatorname{int} V_n \backslash \operatorname{int} 2V_{n+1}$ and $|Au_m - f| = J(u_m) = 2^{-n+1}$. The fact that the approximate solutions u_m $(m \in \mathbb{N})$ are unique is obvious. This completes the proof of Lemma 4.1.2. $\square$

For completeness we will explain the method of construction of the approximate solution u_m. Having solved the system of linear equations (4.1.7), we can find the unique solution of the extremal problem (4.1.6). Therefore, the main difficulty is in finding the number l that satisfies $r = \inf\{J(g);\ g \in G_m\} \in I_l$. Let $p_n^{(m)} = \inf\{p_n(Ag - f);\ g \in G_m\}$. It is well known that $p_n^{(m)} = \sqrt{\dfrac{G(f, Ag_1, \ldots, Ag_m)_n}{G(Ag_1, \ldots, Ag_m)_n}}$. Finding l is equivalent to satisfying the following inequalities:

$$\begin{gathered}
1 < p_1^{(m)} < \infty \ \text{ for } \ r \in \operatorname{int} I_1\,, \\
1 < p_l^{(m)} < 2 \ \text{ for } \ r \in \operatorname{int} I_l,\ l \geq 2\,, \\
p_l^{(m)} \leq 1 \ \text{ and } \ p_{l+1}^{(m)} \geq 2 \ \text{ for } \ r = 2^{-l+1} \ \ (l \in \mathbb{N})\,.
\end{gathered}$$

For $r \in \operatorname{int} I_l$ $(l \in \mathbb{N})$, the fact that these are equivalent follows from the corollary to Lemma 4.1.1. We will prove the equivalence for $r = 2^{-l+1}$ $(l \in \mathbb{N})$. Let $r = \inf\{J(g);\ g \in G_m\} = 2^{-l+1}$. Then, in view of the proximality of the finite dimensional subspace AG_m in the metrizable space E with respect to the metric (3.1.4), for some $g_1 \in G_m$ the equality $J(g_1) = 2^{-l+1}$ holds. Therefore it follows that $p_l(Ag_1 - f) = q_{2^{-l+1}}(Ag_1 - f) \leq 1$ and so $p_l^{(m)} \leq 1$. Now assume that $p_{l+1}^{(m)} < 2$. Then for some $g_2 \in G_m$ we have $p_{l+1}(Ag_2 - f) = p_{l+1}^{(m)} < 2$, that is, $\|Ag_2 - f\|_{l+1} < 2^{-l+1}$. We at once see that $|Ag_2 - f| < 2^{-l+1}$, and this contradicts our assumption.

Now let $p_l^{(m)} \leq 1$ and $p_{l+1}^{(m)} \geq 2$. It follows that for some $g_3 \in G_m$ we have $r = \inf\{J(g);\ g \in G_m\} = |Ag_3 - f| \leq 2^{-l+1}$. If we assume that strict inequality holds here, then we find that $p_{l+1}^{(m)} < 2$, which contradicts our assumption. Thus, to find l, we must verify that the inequalities listed above hold for $p_n^{(m)}$. In some cases, we cannot verify them for all $p_n^{(m)}$ $(1 \leq n \leq l)$. For instance, if we have established that $2^{-n+1} < p_1^{(m)} < 1$ for some $n \in \mathbb{N}$, then it follows from the inequality $2^{n-1}\| \cdot \|_1 \leq p_n(\cdot)$ that $Ag - f \overline{\in} V_n = K_{2^{-n+1}}$ for all $g \in G_m$ and so $2^{-n+1} < r < 1$.

Next, we find $p_{[n/2]}^{(m)}$. If $p_{[n/2]}^{(m)} < 1$, then $2^{-n+1} < r < 2^{-[n/2]+1}$. If, in addition, $p_{[n/2]+1}^{(m)} \geq 2$, then $r = 2^{-[n/2]+1}$. If $p_{[n/2]}^{(m)} > 1$, this implies that $2^{-[n/2]+1} < r < 1$. Next, we consider $p_{[n/4]}^{(m)}$ or $p_{[3n/4]}^{(m)}$. By continuing this process we refine the intervals we have obtained and, finally, we find l such that $r \in \operatorname{int} I_l$ or $p_l^{(m)} \leq 1$ and $p_{l+1}^{(m)} \geq 2$. In the first case $r = 2^{-l+1} p_l^{(m)}$, and in the second $r = 2^{-l+1}$.

Having found the approximate solution $u_m \in G_m$, to find the approximate solution u_{m+1} in G_{m+1}, we verify that the above set of inequalities holds just for $p_n^{(m+1)}$ $(n \geq l)$, and so on.

A natural question arises: does the sequence of approximate solutions $\{u_m\}$ converges to the exact solution u_0 of equation (4.1.1). This is the case for the above class of operators.

Theorem 4.1.3. *Suppose that the topology of the Fréchet space E is induced by a non-decreasing sequence of inner products $\{(\cdot,\cdot)_n\}$, A is an injective operator and u_0 is the exact solution of the equation* (4.1.1). *If the sequence of basis functions $\{g_i\}$ is A-complete and there exists a continuous inverse operator $A^{-1} \in \mathcal{L}(E, G)$, then the sequence of approximate solutions $\{u_m\}$ constructed using the method of least squares converges to u_0 in G. Moreover, the following estimates hold:*

a) *for every n and m,*

$$|u_0 - u_m|_n \leq C_n \|Au_0 - Au_m\|_{\sigma'(n)}\,. \tag{4.1.8}$$

b) *for every n there exists $m_0 = m_0(n)$ such that for every $m > m_0$,*

$$|u_0 - u_m|_n \leq C_n |Au_m - f|\,, \tag{4.1.9}$$

where $|\cdot|$ *is the quasinorm of the metric* (2.5.8).

Proof. Since the sequence $\{g_i\}$ is A-complete, for u_0 and $\varepsilon > 0$ there exist $m_0 = m_0(\varepsilon)$ and $\alpha_1, \ldots, \alpha_{m_0}$ such that for every $m \geq m_0$,

$$|Au_0 - Au_m| \leq \left|Au_0 - \sum_{i=1}^{m_0} \alpha_i A g_i\right| < \varepsilon .$$

It therefore follows that the sequence $\{Au_m\}$ converges to $Au_0 = f$ in E. Thus, for any norm $\|\cdot\|_n$ on E, the sequence $\{\|Au_0 - Au_m\|_n\}$ converges to zero as $m \to 0$. The inequality in a) holds since A is continuously invertible.

Now we will prove part b). For this, we will first show that for every $n \in \mathbb{N}$ there exists $m_0 = m_0(n)$ such that

$$\|Au_0 - Au_m\|_n \leq |Au_0 - Au_m| \ \text{ for all } m_0 . \tag{4.1.10}$$

For given n there exists $m_0 = m_0(n)$ such that $|Au_0 - Au_{m_0}| < \sup I_n$. Let us consider the case when $|Au_0 - Au_{m_0}| \in \operatorname{int} I_n$. Then, using the corollary of Lemma 4.1.1, we have

$$|Au_0 - Au_0| = \|Au_{m_0} - Au_{m_0}\|_n .$$

If $|Au_0 - Au_{m_0}| = 2^{-n+1}$, then, using part b) from the corollary to Lemma 4.1.1, we have

$$\|Au_0 - Au_{m_0}\|_n \leq |Au_0 - Au_{m_0}| .$$

But if $|Au_0 - Au_{m_0}| \in I_{n+1} = [2^{-n}, 2^{-n+1}[$, then

$$\|Au_0 - Au_{m_0}\|_n \leq \|Au_0 - Au_{m_0}\|_{n+1} \leq |Au_0 - Au_{m_0}| .$$

Since $|Au_0 - Au_{m_0}| \in I_p$ for some $p \geq n$, it follows that (4.1.10) holds for $m = m_0$. As is known, the following inequality is true:

$$|Au_0 - Au_{m_0+1}| \leq |Au_0 - Au_{m_0}| .$$

Thus, if we repeat our previous argument, we conclude that (4.1.10) holds for $m = m_0 + 1$. Using a similar line of reasoning we can establish (4.1.10) for any $m \geq m_0$. Further, since A is continuously invertible we find that

$$|u_0 - u_m|_n \leq C_n \|Au_0 - Au_m\|_{\sigma'(n)} \leq C_n |f - Au_m| \ \text{ for } \ m > m_0(\sigma'(n)) ,$$

that is, (4.1.9) holds. $\square$

4.1.3 Application for the approximate solution equation containing tame operators

It should be noted that if the inverse operator is tame, then in inequality (4.1.8) we can set $\sigma'(n) = n + l$, where $l \in \mathbb{N}$ is some number depending on A. If the inverse operator is tamely isometric, i.e. $A^{-1} \in \mathcal{L}_0(R(A), D(A))$, then in inequality (4.1.8) we can set that $\sigma'(n) = n$. If the operator $A^{-1} \in \mathcal{L}_F(R(A), D(A))$, then in inequality (4.1.8) C_n can be chosen independent of n.

For some Fréchet spaces and for some classes of operators, the form of the function σ_T, characterizing the continuity of the linear operator $T \in \mathcal{L}(E)$, and also the form of tame operators are known. Let us give concretization of Theorem 4.1.3 for such spaces and operators:

1) Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a non-decreasing sequence of positive numbers tending to infinity. By $\Lambda_1(\alpha)$ it is denoted the Fréchet space of power series of finite type, i.e. the space of sequences $\xi = \{\xi_i\}$, for which the hilbertian norms

$$\|\xi\|_n^2 = \sum_{j=1}^{\infty} e^{-2\alpha_j/n} |\xi_j|^2, \quad n \in \mathbb{N}\,.$$

And by $\Lambda_\infty(\alpha)$ it is denoted the Fréchet space of power series of infinite type, i.e. the space of sequences $\xi = \{\xi_j\}$, for which the hilbertian norms

$$|\xi|_n^2 = \sum_{j=1}^{\infty} \exp(2n\alpha_j) |\xi_j|^2, \quad n \in \mathbb{N} \tag{4.1.11}$$

are finite. It is known [47] that for each operator $T \in \mathcal{L}(\Lambda_1(\alpha))$ the inequality $\sigma_T(n) \le an$ is true, where $a \in \mathbb{N}$ is some number depending on α, i.e. the space $\Lambda_1(\alpha)$ is tame. From this we obtain that the inequality $\sigma_T(n) \le an$ is true.

Corollary 1. *Let A be an injective operator mapping the Fréchet space $\Lambda_1(\alpha)$ into itself, with a continuous inverse operator A^{-1}. If u_0 is an exact solution of the equation* (4.1.1) *and the sequence of basis functions $\{g_i\}$ from $\Lambda_1(\alpha)$ is A-complete, then the sequence of approximate solutions $\{u_m\}$, constructed by the method of least squares, converges to u_0 in $\Lambda_1(\alpha)$. Moreover, if the image of the operator A is tamely isomorphic to the space $\Lambda_1(\alpha)$, then for each n and m the inequality*

$$\|u_0 - u_m\|_n \le C_n \|Au_0 - Au_m\|_{an+l}$$

holds, where $a \in \mathbb{N}$ depends on α, $l \in \mathbb{N}$ depends on A and $C_n > 0$.

Indeed, the first assertion of Corollary 1 follows from Theorem 4.1.3. Further, let the image of the operator A be tamely isomorphic to the space $\Lambda_1(\alpha)$ and this isomorphism is realized by the mapping $L : R(A) \to L_1(\alpha)$. Then the following commutative diagram holds:

$$\begin{tikzcd} R(A) \arrow[rr, "A^{-1}"] \arrow[dr, "L"'] & & \Lambda_1(\alpha) \\ & \Lambda_1(\alpha) \arrow[ur, "T"'] & \end{tikzcd}$$

where $A^{-1} = T \circ L$ and $T = A^{-1} \circ L^{-1}$ is a continuous mapping of the space $\Lambda_1(\alpha)$ into $\Lambda_1(\alpha)$. By virtue of the above-mentioned theorem from [47] and the tameness of the operator L, we have the estimates

$$\begin{aligned} \|u_0 - u_m\|_n = \|A^{-1}L^{-1}LA(u_0 - u_m)\|_n = \|TLA(u_0 - u_m)\|_n \leq \\ \leq C_{1n}\|LA(u_0 - u_m)\|_{an} \leq C_{1n} \cdot C_{2n}\|A(u_0 - u_m)\|_{an+l} \\ = C_n\|A(u_0 - u_m)\|_{an+l}\,, \end{aligned}$$

where $a \in \mathbb{N}$ depends on α and $l \in \mathbb{N}$ exists due to the tameness of the operator L.

According to ([11], p. 117), nuclear Fréchet spaces are isomorphic to the Fréchet space of finite-type power series if and only if they satisfy the well-known properties $(\underline{D}N)$ and $(\overline{\Omega})$. Moreover, if a nuclear Fréchet space E satisfies the conditions $(\underline{D}N)$ and $(\overline{\Omega})$, then there exists a unique (up to equivalence) sequence $\varepsilon(E)$ called the associated exponential sequence such that if E has the property that E is isomorphic to the space of power series, then E is isomorphic to $\Lambda_1(\varepsilon(E))$ or $\Lambda_\infty(\varepsilon(E))$, depending on the type of the sequence space. From Corollary 1, taking into account what has just been said, we obtain

Corollary 2. *Let E be a Fréchet space with a generating sequence of inner products $\{(\cdot,\cdot)_n\}$, $\|\cdot\|_n = (\cdot,\cdot)_n^{1/2}$ and A be an injective linear operator, mapping the space E into itself, with continuous inverse A^{-1}. If u_0 is an exact solution of equation* (4.1.1) *and the sequence of basis functions $\{g_i\}$ from E is A-complete, then the sequence of approximate solutions $\{u_m\}$, constructed by the least squares method, converges to u_0 in E. Moreover, if E is nuclear, isomorphic to the space $\Lambda_1(\varepsilon(E))$ and the image of the operator A is also tamely isomorphic to the space $\Lambda_1(\varepsilon(E))$, then for each n and m the inequality*

$$\|u_0 - u_m\|_n \leq C_n\|Au_0 - Au_m\|_{an+l}$$

holds, where $a \in \mathbb{N}$ depends on $\varepsilon(E)$, $l \in \mathbb{N}$ depends on A and $C_n > 0$.

Let us now give an example of a Fréchet space satisfying the conditions of Corollary 2. In ([11], Theorem 1.7), it is proved that the Fréchet space of analytic functions $\mathcal{O}(X)$, defined on a Stein manifold X with a known sequence of integral norms, is isomorphic to the Fréchet space of power series of finite type if and only if there exists a bounded plurisubharmonic exhaustive function on X. Such manifolds are called hyperconvex. In particular, as is well known, the space of analytic functions in the unit half-disk $\mathcal{O}(\Delta^d)$ is isomorphic to the space $\Lambda_1(n^{1/d})$. We do not know whether these isomorphisms are tame.

In ([47], Theorem 1.3), it was proved that the family of finite limit points of the set $Q = \{\alpha_j/\alpha_\nu\}_{j,\nu\in\mathbb{N}}$ is bounded if and only if there exists $a \in \mathbb{N}$ depending on α such that for each $T \in \mathcal{L}(\Lambda_\infty(\alpha))$ there exists $l \in \mathbb{N}$ such that the inequality $\sigma_T(n) \leq an + l$ holds. Such spaces of power series of infinite type $\Lambda_\infty(\alpha)$ are tame and the above functions S_k can be chosen as follows: $S_k(n) = a \cdot n + k$ (a can be equated to unity by choosing a sequence equivalent to $\alpha = \{\alpha_j\}$).

Corollary 3. *Let A be an injective, continuously invertible operator, mapping the Fréchet space $\Lambda_\infty(\alpha)$ into itself. If u_0 is an exact solution of equation* (4.1.1) *and the sequence of basis functions $\{g_i\}$ from $\Lambda_\infty(\alpha)$ is A-complete, then the sequence of approximate solutions $\{u_m\}$, constructed by the least-squares method, converges to u_0 in $\Lambda_\infty(\alpha)$. Moreover, if $\Lambda_\infty(\alpha)$ is tame and the image of the operator A is tamely isomorphic to the space $\Lambda_\infty(\alpha)$, then for each n and m the inequality*

$$|u_0 - u_m|_n \leq C_n |Au_0 - Au_m|_{an+l}$$

holds, where $a \in \mathbb{N}$ depends on α, $l \in \mathbb{N}$ depends on A and $C_n > 0$.

Corollary 3 is proved similarly to the proof of Corollary 2, since according to [47] and for the tame space $\Lambda_\infty(\alpha)$ there exists a constant $a \in \mathbb{N}$ such that for every continuous operator $T : \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$ there exists a constant $l \in \mathbb{N}$ and a sequence of numbers C_n such that for all n we have

$$|Tx|_n \leq C_n |x|_{an+l}\,,$$

where the norms $|\cdot|_n$ on $\Lambda_\infty(\alpha)$ are defined by the equality (4.1.11).

According to [11], the Fréchet space of analytic functions $\mathcal{O}(X)$ on a Stein manifold X is isomorphic to the Fréchet space $\Lambda_\infty(\alpha)$ if and only if every bounded plurisubharmonic function on X is constant. In particular, the space of entire functions $\mathcal{O}(C^d)$ is isomorphic to the space $\Lambda_\infty(n^{1/d})$. Therefore, Corollary 3 is valid for some tamely invertible operators mapping spaces $\mathcal{O}(X)$ to themselves or having an image tamely isomorphic to the space $\Lambda_\infty(\alpha)$. We do not know an exact characterization of the Stein manifold X for which the Fréchet space $\mathcal{O}(X)$ is tame, and the form of tame operators from the space $\mathcal{L}(\mathcal{O}(X))$.

Let us give an example of a Fréchet space s of rapidly decreasing sequences for which the form of tame operators is known. The topology of the space s is given by the sequence of hilbertian norms

$$|\xi|_n^2 = \sum_{j=1}^{\infty} j^{2n} |\xi_j|^2, \quad \xi = \{\xi_j\} \in s, \quad n = 0, 1, \dots .$$

For a subspace F of s, F_n denotes the Hilbert space $(\widetilde{F, |\cdot|_n})$ – the completion of the space $(F, |\cdot|_n)$. According to ([47], Theorem 3.1), if T is a tame operator from $\mathcal{L}(s)$ and $F = R(T)$ is the image of the operator T, then there exist a complete orthonormal system $\{e_n\}$ in F_0, consisting of elements of F and an equicontinuous sequence $\{y_n\}$ from s' such that for each $\nu \in \mathbb{N}$ there exists $\mu \in \mathbb{N}$ with the properties

$$\sum_{n=1}^{\infty} |y_n|'_\mu |e_n|_\nu < \infty$$

and

$$Tx = \sum_{n=1}^{\infty} y_n(x) e_n \,, \tag{4.1.12}$$

where $|\cdot|'_\mu$ is the norm dual to the norm $|\cdot|_\mu$.

Corollary 4. *Let A be an injective continuously invertible operator mapping the Fréchet space s to itself. If u_0 is an exact solution of equation* (4.1.1) *and the sequence of basis functions $\{g_i\}$ from s is A-complete, then the sequence of approximate solutions $\{u_m\}$ constructed by the least squares method converges to u_0 in s. Moreover, if the image of the operator A is tamely isomorphic to the space s and A^{-1} has the form* (4.1.12)*, then for each n and m the inequality*

$$|u_0 - u_m|_n \le C_n |Au_0 - Au_m|_{an+l}$$

holds, where $a, l \in \mathbb{N}$ and $C_n > 0$.

Indeed, repeating the arguments that were used to prove Corollary 2, we obtain the following diagram:

$$\begin{tikzcd} R(A) \arrow[rr, "A^{-1}"] \arrow[dr, "L"'] & & s \\ & s \arrow[ur, "T"'] & \end{tikzcd}$$

where $A^{-1} = T \circ L$ and the identity operator $I = L^{-1}L$ is tame. Therefore, by ([47], Theorem 3.1), we have the following estimates:

$$|u_0 - u_m|_n = |A^{-1}L^{-1}LA(u_0 - u_m)|_n \le C_{1n} |L^{-1}LA(u_0 - u_m)|_{an+l_1} \le$$

$$\leq C_{1n} \cdot tC_{2n}|A(u_0 - u_m)|_{an+l_1+l_2} = C_n|A(u_0 - u_m)|_{an+l}\,,$$

where $a, l \in \mathbb{N}$ and $C_n > 0$.

It should be noted that the space s occupies an important place in the structural theory of nuclear Fréchet spaces. In particular, many frequently used Fréchet spaces of infinitely differentiable functions are isomorphic to the space s ([160], p. 602). It is interesting to investigate which of these isomorphisms are tame.

Let us now generalize the concept of a projective process considered in [134] and show that the above generalization of the least squares method is its concrete realization. Let G and E be Fréchet spaces and $\{\mathbb{P}_m\}$ and $\{Q_m\}$ be two sequences (generally speaking, unbounded and nonlinear) of projections with domains $D(\mathbb{P}_m) \subset G$, $D(Q_m) \subset E$ and closed sets of values $R(\mathbb{P}_m) \subset G$, $R(Q_m) \subset E$. Let $A : G \to E$ be a linear operator (in [134] its continuity was required). By the projective process $\{\mathbb{P}_m\,, Q_m\}$ for obtaining an approximate solution of the operator equation (4.1.1) we mean the transition from this equation to the projection equation

$$Q_m A \mathbb{P}_m u = Q_m f,\ f \in D(Q_m) \tag{4.1.10}$$

for which the solution $u_m \in R(\mathbb{P}_m)$ can be found in one way or another.

We will call the projective process $\{\mathbb{P}_m\,, Q_m\}$ applicable to the operator A and write $A \in \Pi\{\mathbb{P}_m, Q_m\}$, if, starting from some n_0, the equation (4.1.10) for each element $f \in E$ has a single solution u_m and the sequence of these solutions as $m \to \infty$ converges to the solution of the equation (4.1.1).

In our case, the operator Q_m is a strong metric projection of the Fréchet space E onto its closed subspace AG_m, that is, an operator that assigns to each element $f \in E$ its unique strong best approximation in AG_m. The operator $\mathbb{P}_m : G \to G_m$ is a projection operator onto a finite-dimensional subspace G_m. This projectve process does not fit into covered by the framework of Banach spaces and has not been discussed before.

4.2 Symmetric and self-adjoint operators in Fréchet–Hilbert spaces

Symmetric and positive definite operators in Hilbert spaces are of most importance for quantum mechanics and mathematical physics and they are studied in detail in the monographs [**?**, 105, 109, 139, 140]. Continuous self-adjoint operators in the case of strict Fréchet–Hilbert spaces were first defined and studied in [84, 85]. Our definitions do not require continuity of the operator A and the results obtained are valid not only for strict Fréchet–Hilbert spaces, but also for countable-Hilbert and nuclear Fréchet spaces.

Let $A \in L_0(E)$, then for each $n \in \mathbb{N}$ we define the projection operator $A_n : (E/\operatorname{Ker}\|\cdot\|_n, \|\widehat{\ }\|_n) \to (E/\operatorname{Ker}\|\cdot\|_n, \|\widehat{\ }\|_n)$ using the equality

$$A_n(K_n x) = K_n(Ax), \quad x \in D(A), \tag{4.2.1}$$

where $K_n : E \to E/\operatorname{Ker}\|\cdot\|_n$ is the canonical map. The condition for the correctness of the operator A_n is the following: for any x_1 and x_2 from E, the condition $x_1 - x_2 \in \operatorname{Ker} K_n$ implies $A_n(x_1 - x_2) \in \operatorname{Ker} K_n$.

Let us prove that $A \in L_0(E)$ if and only if A_n is continuous for each $n \in \mathbb{N}$. Indeed, if $A \in L_0(E)$, then for every $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$\|A_n \widehat{K}_n x\|_n = \|\widehat{K_n A x}\|_n = \|Ax\|_n \le c_n \|x\|_n = c_n \|\widehat{K_n x}\|_n,$$

i.e. A_n is continuous. The converse statement can be proved by similar reasoning and we omit it.

Let E be a Fréchet space with an nondecreasing sequence of Hilbert seminorms $\{\|\cdot\|_n\}$, where $\|x\|_n = (x,x)_n^{1/2}$ for each $x \in E$ and $(\cdot,\cdot)_n$ are the semi-inner product on E. Let A be a linear operator with dense domain $D(A)$. If for $y \in E$ there is an element y^* such that

$$(Ax, y)_n = (x, y^*)_n \tag{4.2.2}$$

for each $x \in D(A)$ and $n \in \mathbb{N}$, then the equality $A^* y = y^*$ defines the operator $A^* : E \to E$, which we call *the Hilbert conjugate of the operator* A. In other words, $D(A^*)$ consists of $y \in E$ for which there exists a vector y^* such that (4.2.2) holds for all $x \in D(A)$ and $n \in \mathbb{N}$. This is equivalent to the fact that the extension of functionals $f_n(x) = (Ax, y)_n$ defined on $D(A)$ to continuous functionals is generated for each $n \in \mathbb{N}$ by the same element $y^* \in E$. Given $y \in D(A^*)$, the element y^* is uniquely determined by the identities (4.2.2). It should also be noted that, as the example below shows, the operator A^* differs from the usual topological adjoint A' and therefore (also keeping in mind [212]) we called it Hilbert.

An operator A with a dense domain $D(A)$ is called *symmetric* if $A \subset A^*$, i.e. if the adjoint operator A^* is a continuation of A.

The symmetric operator A can also be defined in the following equivalent way: the operator A is called symmetric if

$$(Au, v)_n = (u, Av)_n$$

for all $u, v \in D(A)$ and $n \in \mathbb{N}$.

A symmetric operator A is called *self-adjoint* (or hypermaximal according to the terminology introduced by von Neumann [182]) if $A = A^*$. Currently, the first of these terms is more commonly used, and we will also support a similar terminology in the case of Fréchet spaces.

The definition of a self-adjoint operator in the case of a continuous operator A mapping a strict Fréchet–Hilbert space into itself was given in [84], and in [85] a Theorem on the spectral representation of a self-adjoint operator in such spaces was proved.

A symmetric operator A is called *positive definite* in the space E if for every $n \in \mathbb{N}$ there exists $\gamma_n \geq 0$ such that

$$(Ax, x)_n \geq \gamma_n (x, x)_n \quad \text{for every} \quad x \in E.$$

Let again E be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\| \cdot \|_n\}$, $\| \cdot \|_n = (\cdot, \cdot)_n^{1/2}$, $K_n : E \to E/\mathrm{Ker}\, \| \cdot \|_n$ be canonical mapping, $(E/\mathrm{Ker}\, \| \cdot \|_n, \widehat{\|\cdot\|}_n)$ be normed space with the associated norm $\widehat{\|K_n x\|}_n = \|x\|_n$ and the inner product $\langle k_n x, k_n y\rangle_n = (x, y)_n$, $E_n = (\widetilde{E/\mathrm{Ker}\, \| \cdot \|_n}, \widehat{\|\cdot\|}_n)$ be its completion. Let $A : D(A) \subset E \to E$ be a linear operator with dense domain $D(A)$. For each $n \in \mathbb{N}$, we define the linear operator $A_n : E_n \to E_n$ with the dense domain $D(A_n)$ by the equality (4.2.1).

Lemma 4.2.1. *Let E be a Fréchet space with an nondecreasing sequence of hilbertian seminorms $\{\| \cdot \|_n\}$ and $A : D(A) \subset E \to E$ be a linear operator. A is symmetric and positive definite with dense domain $D(A)$ in E if and only if the operators A_n, defined by equalities* (4.2.1)*, are symmetric and positive definite in $(\widetilde{E/\mathrm{Ker}\, \| \cdot \|_n}, \widehat{\|\cdot\|}_n)$ with the dense domains $K_n(D(A))$ for each $n \in \mathbb{N}$.*

Proof. Let the operator A be symmetric and positive definite. From the density $D(A)$ in E we immediately obtain the density $D(A_n) = K_n(D(A))$ in $(E/\mathrm{Ker}\, \| \cdot \|_n, \widehat{\|\cdot\|}_n)$ for each $n \in \mathbb{N}$. Next, for $K_n x, K_n y \in D(A_n)$,

$$\begin{aligned}\langle A_n K_n x, K_n y\rangle_n &= \langle K_n A x, K_n y\rangle_n = (Ax, y)_n \\ &= (x, Ay)_n = \langle K_n x, K_n A y\rangle_n = \langle K_n x, A_n K_n y\rangle_n\end{aligned}$$

and

$$\langle A_n K_n x, K_n x\rangle = \langle K_n A x, K_n x\rangle_n = (Ax, x)_n \geq \gamma_n^2 (x, x) \geq \gamma_n^2 \langle K_n x, K_n x\rangle_n.$$

Let us now assume that A_n is symmetric and positive definite for every $n \in \mathbb{N}$ with dense domain $K_n(D(A))$ and $x, y \in D(A)$. Then we have

$$\begin{aligned}(Ax, y)_n &= \langle K_n A x, K_n y\rangle_n = \langle A_n K_n x, K_n y\rangle_n = \langle K_n x, A_n K_n y\rangle_n \\ &= \langle K_n A x, K_n A y\rangle_n = (x, Ay)_n.\end{aligned}$$

The positive definiteness of the operator A is proved in a similar way. The density of $D(A)$ in E follows from the density of $K_n(D(A))$ in $(E/\widetilde{\operatorname{Ker}\|\cdot\|_n}, \|\cdot\|_n)$ for each $n \in \mathbb{N}$. $\square$

Let us now present a generalization of von Neumann's theorem ([109], p. 121).

Theorem 4.2.2. *Let E be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$. Then the following statements are true:*

a) *The symmetric operator A, defined on the whole space $D(A) = E$, is self-adjoint and continuous, i.e. $A \in L(E)$. Moreover, if E is a strict Fréchet–Hilbert space, then $A \in L_0(E)$, i.e. for every $n \in \mathbb{N}$ there is $c_n > 0$ such that*

$$\|Ax\|_n \leq c_n \|x\|_n \quad \textit{for every} \;\; x \in E. \tag{4.2.3}$$

b) *A symmetric operator A with a dense image $R(A)$ in the space E has an inverse operator A^{-1}, and A^{-1} also is symmetric. Moreover, if A is positive definite, then $A^{-1} \in L_0(E)$.*

c) *Let A be a symmetric operator with dense image $R(A)$ in E. Then A is self-adjoint if and only if A^{-1} is self-adjoint.*

d) *A symmetric operator A whose image $R(A)$ coincides with E is self-adjoint.*

Proof. a) Due to the symmetry of A, we have that $D(A) \subset D^*(A)$, but $D(A) = E$ and therefore $D(A^*) = E$, i.e. $A = A^*$. Let us prove that the operator A is closed. Indeed, let $u_m \to u_0$ and $Au_m \to v_0$ for $m \to \infty$. Then for each $n \in \mathbb{N}$ and $u \in E$ we have that $(v_0, v)_n = \lim_{m\to\infty} (Au_m, v)_n = \lim_{m\to\infty} (u_m, Av)_n = (u_0, Av)_n$, i.e. $v_0 = Au_0$. Thus, the operator A is closed and therefore continuous. If E is a Fréchet–Hilbert space, then the operator A_n defined by (4.2.1) turns out to be defined on the whole Hilbert space $E_n = (E/\widetilde{\operatorname{Ker}\|\cdot\|_n}, \|\widehat{\;}\|)$. Since A_n is symmetric by Lemma 4.2.1, then, by the classical Hellinger–Toeplitz theorem [109], A_n will be continuous for each $n \in \mathbb{N}$, i.e. $A \in L_0(E)$.

b) Let us first note that A^{-1} exists in the case when the operator A is injective, and due to the linearity of A, this occurs only when the equality $Ax = 0$ implies that $x = 0$. Indeed, in our case, for all $y \in D(A)$ and $n \in \mathbb{N}$, we have that $(x, Ay)_n = (Ax, y)_n = 0$. Hence, due to the density of $R(A)$ in E, we obtain that $x \in \operatorname{Ker}\|\cdot\|_n$ for each $n \in \mathbb{N}$, i.e. $x = 0$. This means that there is an inverse operator A^{-1} with a dense domain $D(A^{-1}) = R(A)$ and $A^{-1}Au = u \in D(A)$. Also, $A^{-1}Av = v$ for all $v \in D(A^{-1}) = R(A)$.

Let us now prove that for all $u, v \in D(A^{-1})$ and $n \in \mathbb{N}$ the following equalities are true:

$$(A^{-1}u, v)_n = (u, A^{-1}v)_n\,.$$

Indeed, if $A^{-1}u = x$ and $A^{-1}v = y$, then $u = Ax$ and $v = Ay$. Therefore, due to the symmetry of the operator A, we have that $(A^{-1}u, v)_n = (x, Ay)_n = (Ax, y)_n = (u, A^{-1}v)_n$ for all $n \in \mathbb{N}$, i.e. A^{-1} is symmetric.

Further, from the positive definiteness of the operator A it follows that for every $n \in \mathbb{N}$ there exists $\gamma_n > 0$ such that

$$(Au, u)_n \geq \gamma_n \|u\|_n^2 \,.$$

From the Cauchy–Buniakowski inequality

$$(Au, u)_n \leq \|Au\|_n \|u\|_n$$

it follows that

$$\|Au\|_n \|u\|_n \geq \gamma_n \|u\|_n^2 \,.$$

Hence, for $u \in E$, if $\|u\|_n = 0$, then in the last relation we have equality, and for $\|u\|_n \neq 0$ we have the inequality

$$\|Au\|_n \geq \gamma_n \|u\|_n \,,$$

i.e. $A^{-1} \in L_0(E)$ due to (4.1.4).

c) Let $A = A^*$. In the proof of statement b) it was proven that A^{-1} exists and is symmetric, so there is an adjoint operator $(A^{-1})^*$. It is enough to prove that $(A^{-1})^* = (A^*)^{-1}$, since it immediately follows from the condition that $A^{-1} = (A^*)^{-1}$. If $u \in D(A)$ and $v \in D((A^{-1})^*)$, then for each $n \in \mathbb{N}$ we have

$$(u, v)_n = (A^{-1}Au, v)_n = (Au, (A^{-1})^* v)_n = (u, A^*(A^{-1})^* v)_n \,.$$

This means that $(A^{-1})^* v \in D(A^*)$ and $A^*(A^{-1})^* v = v$. Similarly, if $u \in D(A^{-1})$ and $v \in D(A^*)$, then for each $n \in \mathbb{N}$ we have

$$(u, v)_n = (AA^{-1}u, v)_n = (A^{-1}u, A^* v)_n = (u, (A^{-1})^*(A^* v))_n \,,$$

i.e. $A^* v \in D((A^{-1})^*)$ and $(A^{-1})^*(A^* v) = v$. It follows that $(A^{-1})^* = (A^*)^{-1}$. The converse statement is proved by applying the already proved statement for A^{-1}.

d) By virtue of statement b), there is a symmetric inverse of A^{-1}. But $D(A^{-1}) = R(A) = E$ and therefore, by virtue of a), the operator A^{-1} is self-adjoint, i.e. $A^{-1} = (A^{-1})^*$. But then, by virtue of c), $A = A^*$ and A is self-adjoint. $\square$

It should be noted that statement a) is a generalization of the well-known Hellinger–Toeplitz theorem on the continuity of a symmetric operator defined over the whole space.

The application of Theorem 4.2.2 proves the following generalization of the well-known theorem of Friedrichs, Stone, Vintner ([109], p. 123).

Theorem 4.2.3. *Let E be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$ and $A : D(A) \subset E \to E$ be a symmetric and positive definite operator with a dense image $R(A)$. Then A has a self-adjoint and positive definite extension $\widetilde{A}$ such that $R(\widetilde{A}) = E$.*

Proof. By statement b) of Theorem 4.2.2, there is a symmetric inverse A^{-1} to the operator A, which belongs to $L_0(E)$. Let $B = A^{-1}$. The operator B can be extended by continuity in a unique way to the entire space E. Let us denote this continuation by $\widetilde{B}$. For $x \in E$ it is defined by the equality $\widetilde{B}x = \lim\limits_{k\to\infty} A^{-1}x_k$, where $x_k \in R(A)$ and $\lim\limits_{k\to\infty} x_k = x$. Let us prove that $\widetilde{B}$ is also symmetric. Indeed, let $x, y \in E$ and $\lim\limits_{k\to\infty} x_k = x$ and $\lim\limits_{m\to\infty} y_m = y$, where $x_k, y_m \in R(A)$. Then for each $n \in \mathbb{N}$ we have

$$(\widetilde{B}x, y)_n = \big(\lim_{k\to\infty} A^{-1}x_k, \lim_{m\to\infty} y_m\big)_n = \lim_{k\to\infty}\lim_{m\to\infty}(A^{-1}x_k, y_m)_n =$$
$$= \lim_{k\to\infty}\lim_{m\to\infty}(x_k, A^{-1}y_m)_n = (x, \widetilde{B}y)_n.$$

From statement a) of Theorem 4.2.2 it follows that $\widetilde{B}$ is self-adjoint. Since $\widetilde{B}$ is a continuation of A^{-1}, then $R(\widetilde{B}) \supset R(A^{-1}) = D(A)$, i.e. $R(\widetilde{B})$ is dense everywhere in E. If we now apply statement c) of Theorem 4.2.2, for the operator $\widetilde{B}$ we obtain that $\widetilde{B}^{-1}$ is also self-adjoint. Namely, $\widetilde{B}^{-1}$ is a self-adjoint extension of the operator A. Let us denote $\widetilde{B}^{-1}$ by $\widetilde{A}$. Let us prove that if $x \in D(A)$, then $Ax = \widetilde{A}x$. Indeed, then $Ax \in R(A)$, $A^{-1}Ax = \widetilde{B}Ax = x$, i.e. $Ax = \widetilde{B}^{-1}x$ and therefore $Ax = \widetilde{A}x$. Since $R(\widetilde{A}) = D(\widetilde{B}) = E$, $\widetilde{A}$ satisfies all the requirements of Theorem 4.2.2. □

It is not known whether Theorem 4.2.3 is valid without the requirement $\overline{R(A)} = E$. In the case of Hilbert spaces, this follows from the positive definiteness of the operator A ([47], Theorem X.26, p. 205).

Corollary. *Let E be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$ and $A : E \to E$ be a symmetric and positive definite operator with a dense range $R(A)$. Then A has an extension $\widetilde{A}$ such that the equation*

$$\widetilde{A}u = f \tag{4.2.4}$$

has a unique solution for each $f \in E$.

We will call the solution $u_0 \in D(A)$ of the equation (4.2.4) *classical*, and if $u_0 \notin D(A)$, then *generalized*.

It should also be added that for each $n \in \mathbb{N}$ the extension of the operator $\widetilde{A}$ also satisfies the inequality

$$(\widetilde{A}x, x)_n \geq \gamma_n (x, x)_n.$$

Below we will show that the generalized solution is always contained in the energetic space E_A of the operator A, to the definition of which we proceed.

4.2.1 Energetic space of symmetric and positive definite operators

Let $(E, \mathfrak{T})$ be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, $A : D(A) \subset E \to E$ be a symmetric and positive definite operator. Along with the induced topology, we endow $D(A)$ with the topology $\mathfrak{T}_A$, a generating sequence of hilbertian seminorms

$$[x]_n = [x, x]_n^{1/2}, \quad x \in D(A), \quad n \in \mathbb{N}, \tag{4.2.5}$$

where

$$[x, y]_n = (Ax, y)_n, \quad x \in D(A), \quad n \in \mathbb{N}. \tag{4.2.6}$$

From the positive definiteness of the operator A it follows that the topology $\mathfrak{T}_A$ is not weaker than the topology $\mathfrak{T} \cap D(A)$. Let E_A denote the completion of $D(A)$ in the topology $\mathfrak{T}_A$ and call it the energetic space of the operator A. We will call the quantities $[x, y]_n$ and $[x]_n$ *energetic semi-inner product* $x, y \in D(A)$ and *energetic seminorms* $x \in D(A)$, respectively. By virtue of the above, we also have that $D(A) \subset E_A \subset E$. Therefore, after $D(A)$ is replenished, elements not contained in $D(A)$ appear in E_A. Therefore, the representations (4.2.5) and (4.2.6) do not hold for all $x, y \in E_A$. Although, for $x, y \in D(\widetilde{A})$ the representations are still valid

$$[x, y]_n = (\widetilde{A}x, y)_n\,. \tag{4.2.7}$$

If we assume that the sequence of seminorms $\{[\,\cdot\,]_n\}$ is non-decreasing, then the Fréchet space $(E_A, \mathfrak{T}_A)$ can be represented as the projective limit of a sequence of Hilbert spaces $\{(\widehat{E_A/\operatorname{Ker}[\,\cdot\,]_n}, \widehat{[\,\cdot\,]}_n)\}$ with respect to the mappings π_{nm} $(m \geq n)$, where $[\widehat{K_{A,n}x}]_n = [x]_n$ is the associated norm on the quotient space $E_A/\operatorname{Ker}[\,\cdot\,]_n$, $K_{A,n} : E_A \to E_A/\operatorname{Ker}[\,\cdot\,]_n$ is canonical mapping, $E_{A,n} = (\widehat{E_A/\operatorname{Ker}[\,\cdot\,]_n}, \widehat{[\,\cdot\,]}_n)$ is completion of the space $(E_A/\operatorname{Ker}[\,\cdot\,]_n, \widehat{[\,\cdot\,]}_n)$, and $\widetilde{\pi}_{A,nm}$ is a continuous extension to $E_{A,m}$ of the canonical map

$$\pi_{A,nm} : (E_A/\operatorname{Ker}[\,\cdot\,]_m, \widehat{[\,\cdot\,]}_m) \to (E_A/\operatorname{Ker}[\,\cdot\,]_n, \widehat{[\,\cdot\,]}_n) \quad (m \geq n).$$

By Lemma 4.2.1, if A is symmetric and positive definite, then A_n is symmetric and positive definite, and therefore on $D(A_n)$ we can define the energetic norm of the operator A_n by the equality

$$[K_n x]_{A_n} = \langle A_n K_n x, K_n x\rangle_n^{1/2}, \quad K_n x \in D(A_n), \tag{4.2.8}$$

and the inner product

$$[K_n x, K_n y]_{A_n} = \langle A_n K_n x, K_n y\rangle_n\,, \quad K_n y, K_n x \in D(A_n). \tag{4.2.9}$$

The completion of $D(A_n)$ by the norm $[\,\cdot\,]_{A_n}$ is the energy space of the operator A_n and is denoted by H_{A_n}.

Theorem 4.2.4. *Let E be a Fréchet space with a generating nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, $A : D(A) \subset E \to E$ be a symmetric and positive definite operator, E_A be the energetic space with an nondecreasing sequence of seminorms* (4.2.5)*. Then the space E_A is represented as the projective limit of a sequence of Hilbert spaces H_{A_n}, energetic spaces of the operator A_n.*

Proof. It is enough to show that the Hilbert spaces $(E_{A,n}, [\widehat{\cdot}]_n)$ and $(H_{A,n}, [\,\cdot\,]_{A_n})$ are isometric. Indeed, since $D(A)$ is everywhere dense in $(E_A, \mathfrak{T}_A)$ and $K_{n,A}(E_A)$ is dense in $(E_{A,n}, [\widehat{\cdot}]_n)$, then $K_{n,A}(D(A))$ is dense in $E_{A,n}$ for every $n \in \mathbb{N}$. Indeed, let n be fixed, $\varepsilon > 0$ and $\widehat{x} \in E_{A,n}$, then there exists $x \in E_A$ such that $[\widehat{x} - K_{n,A}]_n < \frac{\varepsilon}{2}$. Therefore, for $x \in E_A$ and the specified n, there exists $y \in D(A)$ such that $[x - y]_n < \frac{\varepsilon}{2}$, i.e. $[\widehat{K_{n,A}x - K_{n,A}}y]_n < \frac{\varepsilon}{2}$. From here we get that $[\widehat{x} - K_{n,A}y]_n < \varepsilon$. Further, from the definition of the norms $[\widehat{\cdot}]_n$ and $[\,\cdot\,]_{A_n}$ it follows that the normed spaces $(K_{n,A}(D(A)), [\widehat{\cdot}]_n)$ and $(D(A_n), [\,\cdot\,]_{A_n})$ are isometric and this isometry is realized by the mapping $K_{n,A}x \to K_n x$, $x \in D(A)$. Really,

$$[\widehat{K_{n,A}x}]_n = [x]_n = (Ax, x)_n = \langle K_n Ax, K_n x\rangle_n = \langle A_n K_n x, K_n x\rangle_n = [K_n x]_{A_n}\,.$$

Since $K_{n,A}(D(A))$ and $D(A_n)$ are dense in the spaces $E_{A,n}$ and H_{A_n}, respectively, then their completions $(E_{A,n}, [\widehat{\cdot}]_n)$ and $(H_{A_n}, [\,\cdot\,]_{A_n})$ are also isometrical. Hence, in view of the above, it turns out that the space $(E_A, \mathfrak{T}_A)$ is represented as the projective limit of the sequence of Hilbert spaces $\{(H_{A_n}, [\,\cdot\,]_{A_n})\}$.

To illustrate our reasoning, we present a diagram in which the arrows indicate the mappings we mentioned:

$$\begin{tikzcd}
 & E_A \arrow[r, "K_{A,n}"] \arrow[d, "\widetilde{A}"] & (E_{A,n}, [\widehat{\ }]_n) \arrow[d, leftrightarrow] \\
E \arrow[r, "A"] \arrow[d, "K_n"'] & E \arrow[d, "K_n"'] & (E_{A,n}, [\widehat{\ }]_n) \arrow[d, leftrightarrow] \\
E_n = (E/\widetilde{\mathrm{Ker}\,\|\cdot\|_n}, \|\widehat{\cdot}\|_n) \arrow[r, "A_n"] & E_n & (H_{A_n}, [\,\cdot\,]_{A_n}) \arrow[l]
\end{tikzcd}$$

The operators $\widetilde{A}$ and $\widetilde{A}_n$ are surjective and exist by virtue of Theorem 4.2.3. $\square$

4.2.2 Examples of symmetric and self-adjoint operators in Fréchet–Hilbert spaces

1. Position operator in quantum mechanics. The position operator in quantum mechanics is well known, defined by the equality

$$Tx(t) = t \cdot x(t), \tag{4.2.10}$$

where $D(T)$ consists of all complex valued functions $x \in L^2(\mathbb{R})$ for which $t \cdot x(t) \in L^2(\mathbb{R})$. We can define the extension of this operator by the same equality to the space $L^2_{loc}(\mathbb{R})$. The topology of the space $L^2_{loc}(\mathbb{R})$ is generated by a sequence of semi-inner products

$$(x,y)_n = \int\limits_{-n}^{n} x(t)\overline{y(t)}\,dt, \quad x,y \in L^2_{loc}(\mathbb{R}). \tag{4.2.11}$$

Let us prove that the operator (4.2.10) with domain $D(T) = L^2_{loc}(\mathbb{R})$ is a continuous self-adjoint operator. Indeed, for each $n \in \mathbb{N}$ and $x, y \in L^2_{loc}(\mathbb{R})$, we have

$$(Tx,y)_n = \int\limits_{-n}^{n} tx(t)\overline{y(t)}\,dt = \int\limits_{-n}^{n} x(t)\overline{ty(t)}\,dt = (x,Ty)_n\,.$$

By virtue of statement a) of Theorem 4.2.2, it turns out that T is a continuous operator of class $L_0(L^2_{loc}(\mathbb{R}))$, since the space $L^2_{loc}(R)$ is a strict Fréchet–Hilbert space.

2. Taking into account the notation of the previous example, we define an operator on the space $L^2_{loc}(\mathbb{R})$ using the equality

$$Tx(t) = (1+t^2)x(t). \tag{4.2.12}$$

By statement a) of Theorem 4.2.2, this operator is self-adjoint and continuous of class $L_0(L^2_{loc}(\mathbb{R}))$, since $D(T) = L^2_{loc}(R)$ and for each $n \in \mathbb{N}$ and $x, y \in L^2_{loc}(\mathbb{R})$ the following equalities are valid:

$$(Tx, y)_n = \int_{-n}^{n} (1+t^2)x(t)\overline{y(t)}\, dt = \int_{-n}^{n} x(t)\overline{(1+t^2)y(t)}\, dt = (x, Ty)_n\,.$$

The operator (4.2.12) is positive definite, since for each $x \in L^2_{loc}(\mathbb{R})$ and $n \in \mathbb{N}$, the following inequalities hold:

$$\begin{aligned}(Tx, x)_n &= \int_{-n}^{n} (1+t^2)|x(t)|^2\, dt \\ &\geq \min\{(1+t^2);\ t \in [-n, n]\} \int_{-n}^{n} |x(t)|^2\, dt \geq (x, x)_n\,.\end{aligned}$$

On the other hand, $(Tx, x)_n \leq \max\{(1+t^2);\ t \in [-n, n]\}(x, x)_n \leq (1+n^2)(x, x)_n$, i.e. the energy space of the operator (4.2.12) coincides with the space $L^2_{loc}(\mathbb{R})$. It is easy to verify that there is an inverse operator defined by the equality

$$T^{-1}x(t) = \frac{x(t)}{1+t^2}\,.$$

This operator is also self-adjoint, positive definite and continuous of class $L(L^2_{loc}(\mathbb{R}))$. Therefore, T is isometrically tame. Since T is a topological isomorphism, the topological adjoint map $T' : (L^2_{loc}(\mathbb{R}))' \to (L^2_{loc}(\mathbb{R}))'$ is also a strong isomorphism. But the space $(L^2_{loc}(\mathbb{R}))' = L^2_0(\mathbb{R})$ in the strong topology is a strict (LH)-space, which is everywhere dense in the space $L^2_{loc}(R)$. It follows that T' and T^* are different from each other.

3. Let $\Omega \subset \mathbb{R}^l$ be an open set. The space $W^{2,\infty}(\Omega)$ consists of functions f that have generalized derivatives of all orders $f^{(\alpha)} \in L^2(\Omega)$ ($\alpha = (\alpha_1, \dots, \alpha_l)$ is multiindex). The space $W^{2,\infty}(\Omega)$ is considered with the topology of L^2-convergence of derivatives of all orders. This topology is non-normable, but metrizable and is given by an nondecreasing sequence of hilbertian norms

$$\|f\|_{2,n} = \Bigg(\sum_{|\alpha|\leq n-1} \|f^{(\alpha)}\|_2^2\Bigg)^{1/2}, \quad |\alpha| = \sum_{j=1}^{l} \alpha_j, \quad n \in \mathbb{N}, \tag{4.2.13}$$

where

$$\|f\|_2 = \Bigg(\int_{\Omega} |f(t)|^2 dt\Bigg)^{1/2}.$$

It is obvious that the norm (4.2.13) is generated by the scalar product

$$(f,g)_{2,n} = \sum_{|\alpha|\le n-1} (f^{(\alpha)}, g^{(\alpha)})_2, \tag{4.2.14}$$

where $(\cdot,\cdot)_2$ is the scalar product of the space $L^2(\Omega)$.

In [200], it was proven that the space $W^{2,\infty}(\mathbb{R}^l)$ is a complete countable Hilbert space. It is known that $W^{2,\infty}(\mathbb{R}^l) = \bigcap\limits_{n=0}^{\infty} W_2^n(\mathbb{R}^l)$, where $W_2^n(\mathbb{R}^l)$ is a Sobolev space of n-th order. It also follows from Sobolev's theorem that $W^{2,\infty}(\Omega)$ consists of functions having ordinary derivatives of all orders. The space $W^{2,\infty}(\mathbb{R}^l)$ is not nuclear, since it has an infinite-dimensional normable subspace $\mathfrak{M}_{\nu 2}(\mathbb{R}^l)$ (Section 2.3), which consists of entire functions of exponential type ν, whose restrictions to $\mathbb{R}^l$ belong to the space $L^2(\mathbb{R}^l)$.

Using the equality

$$Ax(t) = \frac{i}{h}\left[x\left(t+\frac{h}{2}\right) - x\left(t-\frac{h}{2}\right)\right], \tag{4.2.15}$$

let us define a difference operator, where $h \in R$, $h \neq 0$. Let us prove that this operator is self-adjoint on the space $W^{2,\infty}(\mathbb{R})$. Indeed, for $x, y \in W^{2,\infty}(\mathbb{R})$, we have

$$\begin{aligned}
(Ax,y)_{2,1} &= \frac{i}{h}\int\limits_{-\infty}^{\infty} x\left(t+\frac{h}{2}\right)\overline{y(t)}\,dt - \frac{i}{h}\int\limits_{-\infty}^{\infty} x\left(t-\frac{h}{2}\right)\overline{y(t)}\,dt\\
&\left(t+\frac{h}{2}=x,\ \ t=s-\frac{h}{2},\ \ ds=dt,\ \ t-\frac{h}{2}=s,\ \ t=s+\frac{h}{2}\right)\\
&= \frac{i}{h}\int\limits_{-\infty}^{\infty} x(s)\overline{y\left(s-\frac{h}{2}\right)}\,ds - \frac{i}{h}\int\limits_{-\infty}^{\infty} x(s)\overline{y\left(s+\frac{h}{2}\right)}\,ds\\
&= \frac{i}{h}\int\limits_{-\infty}^{\infty} x(s)\left(\overline{y\left(s-\frac{h}{2}\right)} - \overline{y\left(s+\frac{h}{2}\right)}\right)ds = \frac{i}{h}\int\limits_{-\infty}^{\infty} x(s)\overline{Ay(s)}\,ds\\
&= (x, Ay)_1 .
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
(Ax,y)_{2,2} &= (Ax,y)_{2,1} + ((Ax)', y')_{2,1} = (x,Ay)_{2,1} + (Ax', y')_{2,1} =\\
&= (x,Ay)_{2,1} + (x', Ay')_{2,1} = (x,Ay)_{2,2} .
\end{aligned}$$

Similar reasoning will prove the equality

$$(Ax, y)_{2,n} = (x, Ay)_{2,n}$$

for $n \geq 2$. Since $D(A) = W^{2,\infty}(\mathbb{R})$, then, by statement a) of Theorem 4.2.2, A is a continuous self-adjoint operator.

4. Momentum operator in quantum mechanics. The momentum operator in quantum mechanics is defined by the equality

$$Tx(t) = ix'(t),$$

where i is a complex unit, and $D(T)$ consists of a set of absolutely continuous functions $x(t)$ on $[-1, 1]$, having derivatives $x'(t) \in L^2[-1.1]$. By analogy with Example 1, we define the extension $\widetilde{T}$ of this operator by the equality just given to $L^2_{loc}]a, b[$. Let $a_n \downarrow a$ and $b_n \uparrow b$. We define the topology of the space $L^2_{loc}]a, b[$ by a sequence of hilbertian seminorms

$$\|x\|_n = \left(\int\limits_{a_n}^{b_n} |x(t)|^2 \, dt \right)^{1/2}.$$

We assume that $D(T)$ consists of all functions $x(t) \in L^2_{loc}[a, b]$ that are absolutely continuous on each compact interval in $]a, b[$, i.e. $x'(t) \in L^2_{loc}[a, b]$ and $x(a_n) = x(b_n) = 0$ for each $n \in \mathbb{N}$. By an insignificant change in the reasoning, used to prove the density of the domain of definition of the operator T in the case of the Hilbert space $L^2(\mathbb{R})$ ([71], p. 276), one can also prove that $D(\widetilde{T})$ is everywhere dense in $L^2_{loc}]a, b[$. Applying the results of the example given in ([71], p. 284), we obtain that $\widetilde{T}^* = ix'(t)$, where $D(\widetilde{T}^*)$ is an space of absolute continuous on each compact set $]a, b[$ functions $x(t)$ for which $x'(t) \in L^2_{loc}[a, b]$. Therefore, $D(\widetilde{T}) \subset D(\widetilde{T}^*)$ and $\widetilde{T}$ is an example of a symmetric but not self-adjoint operator.

5. Let H^2 be the space of functions analytic in the unit disk such that their restrictions on the circle $f(\theta) = f(e^{\theta})$ belong to the space $L^2[-\pi, \pi]$, periodic functions summable by square. It is well known that the Fourier coefficients of such functions with negative indices are equal to zero, i.e.

$$\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(\theta) e^{ij\theta} \, d\theta = 0, \quad j = 1, 2, \ldots .$$

Consequently, the expansion of the function $f \in H^2$ in the form of a Fourier series has the form

$$f = \sum_{j=1}^{\infty} (f, u_j) u_j, \quad \text{where } \; u_j = e^{ij\theta} .$$

We denote by $H^{2,\infty}[-\pi,\pi]$ the space of functions analytic in the unit disk such that $f(\theta) = f(e^{1\theta})$ together with their derivatives belong to the space $L^2[-\pi,\pi]$. Let us define the topology of the space $H^{2,\infty}$ by means of a sequence of hilbertian norms

$$\|f\|_{2,n} = \left(\sum_{\alpha=0}^{n-1} \|f^{(\alpha)}\|_2^2\right)^{1/2}, \tag{4.2.16}$$

where

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta\right)^{1/2}.$$

Consider a linear, positive definite operator A with the spectrum $0 < \lambda_1 < \cdots < \lambda_n < \cdots$ and $\lim \lambda_k = \infty$. Then the operator A has the form

$$Af = \sum_{j=1}^{\infty} \lambda_j (f, u_j) u_j. \tag{4.2.17}$$

$D(A)$ consists of all functions f for which the series (4.2.17) converges. Since $D(A)$ contains all finite sums, then $D(A)$ is dense in $H^{2,\infty}[-\pi,\pi]$. Let us prove that for each $n \in \mathbb{N}$, $f, g \in D(A)$, the following equalities hold:

$$(Af, \phi)_{2,n} = (f, A\phi)_{2,n}\,. \tag{4.2.18}$$

We have

$$\begin{aligned}(Af,\phi)_{2,1} &= \left(\sum_{j=1}^{\infty} \lambda_j (f,u_j)u_j, \phi\right) = \sum_{j=1}^{\infty} \lambda_j (f,u_j)(u_j,\phi) \\ &= \sum_{j=1}^{\infty} (f,u_j)\,\overline{\lambda_j(\phi,u_j)} = \left(f, \sum_{j=1}^{\infty} \lambda_j(\phi,u_j)u_j\right) = (f, A\phi)_{2,1}\,.\end{aligned}$$

Further, due to the fact that the derivative of the Fourier series of the function $f \in H^{2,\infty}[-\pi,\pi]$ is the Fourier series of the derivative of the function, we have

$$\begin{aligned}(Af,\phi)_{2,2} &= (Af,\phi) + ((Af)',\phi') = (f,A\phi) + (Af',\phi') = (f,A\phi) + (f'A\phi') \\ &= (f,A\phi) + (f',(A\phi)') = (f,A\phi)_{2,2}\,.\end{aligned}$$

Equalities (4.2.18) are proved similarly for other $n \geq 2$.

The operator (4.2.17) is also positive definite. Indeed, for $f \in D(A)$ we have

$$(Af,f)_{2,1} = \sum_{j=1}^{\infty} \lambda_j (f,u_j)(u_j,f) = \sum_{j=1}^{\infty} \lambda_j |(f,u_j)|^2$$

$$\geq \lambda_1 \sum_{j=1}^{\infty} |(f, u_j)|^2 = \lambda_1 (f, f)_{2,1} .$$

By similar reasoning, we obtain the inequality

$$(Af, f)_{2,n} \geq \lambda_1 (f, f)_{2,n}$$

for other $n \geq 2$, too.

The topology of the energetic space E_A of the operator A, defined by the equality (4.2.17), is generated by the sequence of norms $[f]_n = (Af, f)_{2,n}^{1/2}$.

The inverse operator $A^{-1} f = \sum\limits_{j=1}^{n} \frac{(f,u_j)}{\lambda_j} u_j$ is a continuous operator of class $L_F(H^{2,\infty})$, positive definite and symmetric.

6. Let s be the space of rapidly decreasing sequences with a sequence of scalar products

$$(x, y)_n = \sum_{j=1}^{\infty} j^{2n} x_k y_k, \quad \text{where } \ x = \{x_k\},\ y = \{y_k\} \in s.$$

Consider the operator

$$Ax = (x_1, 2x_2, \ldots, nx_n, \ldots).$$

We have

$$|Ax|_n = \sum_{j=1}^{n} j^{2n} j^2 |x_j|^2 = \sum_{j=1}^{n} j^{2n+2} |x_j|^2 = |x|_{n+1} . \tag{4.2.19}$$

Therefore, A is a tame $(\ell = 1)$ continuous (despite the fact that this definer is not continuous in the Hilbert space ℓ^2).

Next, for each $n \in \mathbb{N}$ and $x, y \in s$ we have

$$(Ax, y)_n = \sum_{j=1}^{\infty} j^{2n+2} x_j y_j = (x, Ay)_n,$$

i.e. A is symmetric and, by virtue of its definition on the whole space s and statement a) of Theorem 4.2.2, is self-adjoint.

For $n \in \mathbb{N}$ and $x \in s$, we also have that

$$(Ax, x)_n = \sum_{j=1}^{\infty} j^{2n+2} |x_j|^2 \geq \sum_{j=1}^{n} j^{2n} |x_j|^2 \geq (x, x)_n , \tag{4.2.20}$$

i.e. A is positive definite.

Since $R(A) = s$, there exists $A^{-1}x = (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots)$ and by (4.2.19), A is continuous of class $L_F(s)$, self-adjoint and positive definite.

By virtue of (4.2.19)m, we also obtain that the energetic space of the operator (4.2.18) again coincides with s.

7. The Schwarz space of rapidly decreasing functions $S(\mathbb{R})$ (see Section 2.6) is the set of infinitely differentiable complex-valued functions $\phi(x)$ on $\mathbb{R}$ for which

$$\|\phi\|_n = \sup\left\{|x^n D^{(n)}\phi(x)|;\ x \in R\right\} < \infty, \quad n \in \mathbb{N}. \tag{4.2.21}$$

Thus, the functions from the space $S(\mathbb{R})$ are those functions that, together with their derivatives, decrease to infinity faster than any polynomial. With the sequence of norms (4.2.21) it is a Fréchet space ([139], p. 152). Let us now present an equivalent sequence of hilbertian norms on $S(\mathbb{R})$

$$\|\phi\|_{n,2} = \left(\sum_{k\le n-1} \int\limits_R |x^k \phi^{(k)}(x)|^2 dx\right)^{1/2}, \quad \phi \in S(\mathbb{R}), \quad n \in \mathbb{N}. \tag{4.2.22}$$

In ([139], p. 161), the following increasing sequence of norms $\{\|\cdot\|_n\}$ is considered, equivalent to (4.2.21). Let $A : S(\mathbb{R}) \to S(\mathbb{R})$ and $A^+ : S(\mathbb{R}) \to S(\mathbb{R})$ be the mappings defined respectively by the equalities

$$A = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \quad A^+ = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \quad N = A^+A.$$

Then we define the hilbertian norms $\|\cdot\|_n$ on $S(\mathbb{R})$ as follows:

$$\|f\|_n = \|(N+1)^n f\|_2, \quad f \in S(\mathbb{R}) \quad \text{and} \quad n \in \mathbb{N}.$$

It is known ([64], see also [160], p. 606) that $S(\mathbb{R})$ is a nuclear countable Hilbert space.

In the next section we will introduce a new sequence of norms equivalent to (4.2.21), which plays an important role in finding approximate solutions for an equation containing a quantum harmonic oscillator operator.

4.2.3 Formation of symmetric operators in Fréchet–Hilbert space

Let T be a closed linear operator mapping an everywhere dense subset $D(T)$ of the Hilbert space H into itself. Von Neumann's theorems on the representation of the orthogonal complement in $H \times H$ of the graph $G(T)$, as well as the symmetry and self-adjointness of the operators TT^*, T^*T, $I + TT^*$ and $I + T^*T$, are well known.

In this section, we generalize these von Neumann results to the case of Fréchet spaces E whose topology is generated by an increasing sequence of hilbertian seminorms $\|x\|_n = (x,x)_n^{1/2}$. Then $E \times E$ is again a Fréchet space whose topology is generated by an increasing sequence of hilbertian seminorms

$$|\{x,y\}|_n = \left(\|x\|_n^2 + \|y\|_n^2\right)^{1/2}, \quad \{x,y\} \in E \times E, \tag{4.2.23}$$

and these seminorms are generated by semi-inner products

$$(\{x,y\},\{w,z\})_n = (x,w)_n + (y,z)_n, \tag{4.2.24}$$
$$\{x,y\}, \{w,z\} \in E \times E, \quad n \in \mathbb{N}.$$

Let again $A : E \to E$ be a closed linear operator and $G(A) = \{\{x, Ax\} \in E \times E;\ x \in D(A)\}$ be a graph of operator A. Let us define a mapping V on $E \times E$ using the equality

$$V\{x,y\} = \{-y, x\}.$$

Proposition 4.2.5. *Let E be a Fréchet space with an increasing sequence of hilbertian seminorms $\|\cdot\|_n = (\cdot,\cdot)_n^{1/2}$ and let $A : E \to E$ be a closed linear operator with the dense domain $D(A)$. If there is a adjoint operator $A^* : E \to E$, then the equality $G(A^*) = \bigcap\limits_{n\in\mathbb{N}} V(G(A))_n^{\perp}$ holds.*

Proof. Let $\{z,w\} \in \bigcap\limits_{n\in\mathbb{N}} V(G(A))_n^{\perp}$. Then for each $(x,y) \in VG(A)$ and $n \in \mathbb{N}$, the following equalities hold:

$$(\{z,w\},\{x,y\})_n = 0.$$

By condition, $\{x,y\} = V(\{x_1,y_1\})$, where $\{x_1,y_1\} \in G(A)$. Therefore, $y_1 = Ax_1$ and $\{x,y\} = V(\{x_1,Ax_1\}) = \{-Ax_1,x_1\}$. Then, for each $n \in \mathbb{N}$, the following equalities are true:

$$(\{z,w\},\ \{-Ax_1,x_1\})_n = -(z,Ax_1)_n + (w,x_1)_n = 0,$$

i.e.

$$(Ax_1,z) = (x_1,w)_n.$$

This is equivalent to the fact that $z \in D(A^*)$ and $w = A^*z$, i.e. $\{z,w\} \in G(A^*)$.

Let us now prove the converse inclusion. Let $\{z,w\} \in G(A^*)$, i.e. $w = A^*z$. Then for an arbitrary $\{-Ax_1,x_1\} \in VG(A)$, where $x_1 \in D(A)$, it is necessary to prove that

$$\{z, A^*z\} \perp \{-Ax_1, x_1\}.$$

This follows from the equalities

$$-(z, Ax_1)_n + (A^*z, x_1)_n = 0,$$

which for each $n \in \mathbb{N}$ follow from the conditions. $\square$

Corollary. *Under the conditions of Proposition* 4.2.5*, the Hilbert adjoint operator is closed.*

It should be noted that, in contrast to Hilbert spaces, it cannot be argued that $V(G(A))$ is the orthogonal complement of $G(A^*)$, since not every closed subspace of the Fréchet space $E \times E$ has an orthogonal complement (see Section 2.4).

Let us now present a sufficient condition for $VG(A)$ to have an orthogonal complement in $E \times E$.

Theorem 4.2.6. *Let $E \times E$ be a Fréchet–Hilbert space* (*resp. countable-Hilbert space*) *with a sequence of hilbertian seminorms* (*resp. hilbertian norms*) (4.2.23) *and $A : E \to E$ is a closed linear operator that has the Hilbert adjoint operator A^*. If a closed subspace $V(G(A))$ has the property (H)* (*resp. the property (C)*) *in $E \times E$, then $E \times E = V(G(A)) \oplus G(A^*)$.*

Proof. Since $V(G(A))$ has orthogonal complement, then, by Proposition 4.2.5, it turns out that $VG(A)^{\perp} = G(A^*)$. Indeed, let $E \times E = VG(A) \oplus V(G(A))^{\perp}$, i.e. each element of $\{z, w\} \in VG(A)^{\perp}$ is orthogonal to an arbitrary element of the subspace $V(G(A))$, i.e. $\{z, w\} \perp \{-Ax_1, x_1\}$. This means that for each $n \in \mathbb{N}$ the equality

$$(z, Ax_1)_n = (w, x_1)_n$$

holds, i.e.

$$(Ax_1, z)_n = (x_1, w)_n.$$

But then $w = A^*z$ and $\{z, w\} \in G(A^*)$. $\square$

Corollary. *Let the Fréchet space $E \times E$ with a sequence of hilbertian seminorms* (4.2.23) *be represented as the sum of its closed subspaces $G(A^*)$ and $VG(A))$. If $D(AA^*)$ and $D(A^*A)$ are dense everywhere in E and $A^{**} = A$, then the operators AA^* and A^*A are symmetric and only positive, and the operators $I + AA^*$ and $I + A^*A$ are self-adjoint, positive definite and have self-adjoint and continuous inverses of class $L_0(E)$.*

Proof. Let's check that AA^* and A^*A are symmetric and only positive. Indeed, let $x, y \in D(AA^*) = \{z \in D(A^*); A^*z \in D(A)\}$, then for each $n \in \mathbb{N}$,

$$(AA^*x, y)_n = (A^*x, A^*y)_n = (x, A^{**}A^*y)_n = (x, AA^*y)_n$$

and

$$(AA^*x, x)_n = (A^*x, A^*x)_n \geq 0.$$

The statement for A^*A is proved similarly.

Let now $x, y \in D(I + AA^*)$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned}((I + AA^*)x, y)_n &= (x + AA^*x, y)_n = (x, y)_n + (AA^*x, y)_n \\ &= (x, y)_n + (x, AA^*y)_n = (x, (I + AA^*)y)_n.\end{aligned}$$

The symmetry of the operator $I + AA^*$ is proved in a similar way. Further, for arbitrary $x \in D(I + AA^*)$ and each $n \in \mathbb{N}$, we have that

$$((I + AA^*)x, x)_n = (x, x)_n + (AA^*x, x)_n = (x, x)_n + (A^*x, A^*x)_n \geq (x, x)_n,$$

i.e. $I + AA^*$ is a positive definite operator.

From the equality $E \times E = VG(A) \oplus G(A^*)$ it follows that for an arbitrary $\{f, g\} \in E \times E$ there are only elements $\{-Ax, x\} \in V(G(A))$ and $\{y, A^*y\} \in G(A^*)$ such that

$$\{f, g\} = \{-Ax, x\} + \{y, A^*y\},$$

or, if we rewrite them in coordinates, then

$$\begin{cases} f = -Ax + y, \\ g = x + A^*y, \end{cases} \tag{4.2.25}$$

in this case $(\{-Ax, x\}, \{y, A^*y\})_n = 0$ for each $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}|\{f, g\}|_n^2 &= \|f\|_n^2 + \|g\|_n^2 = |\{-Ax, x\}|_n^2 + |\{y, A^*y\}|_n^2 \\ &= \|Ax\|_n^2 + \|x\|_n^2 + \|y\|_n^2 + \|A^*y\|_n^2.\end{aligned}$$

In the case $g = 0$, from (4.2.25) we obtain the system

$$\begin{cases} f = -Ax + y, \\ 0 = x + A^*y. \end{cases}$$

It follows that for any $f \in E$ there is $g \in D(AA^*)$ such that $f = y + AA^*y$, i.e. for any $f \in E$ there is a unique y that satisfies the equation

$$(I + AA^*)u = f.$$

This means that $R(I + AA^*) = E$ and therefore, by statement d) of Theorem 4.2.2, the operator $I + AA^*$ is self-adjoint. Again, by virtue of statements b) and c) of Theorem 4.2.2, we obtain that the operator $(I + AA^*)^{-1}$ is also self-adjoint and $(I + AA^*)^{-1} \in L_0(E)$.

The statement for the operator $I + A^*A$ is proved in a similar way. $\square$

Note that the above results for the operator (4.2.12) can be obtained by applying the corollary of Theorem 4.2.6 to the operator (4.2.10), i.e. if we assume that $Tx(t) = tx(t)$, then $(I + TT^*)x(t) = (1 + t^2)x(t)$.

4.2.4 Extension of some symmetric operators from Hilbert spaces to strict Fréchet–Hilbert spaces

We present here a sufficient condition for symmetric and self-adjoint operators defined in a Hilbert space to have the same extensions in some strict Fréchet–Hilbert spaces. In this case, their domain of definition significantly expands and in some cases these extensions turn out to be continuous, despite the fact that the operators themselves are unbounded in the original Hilbert spaces.

For this purpose, we will apply Theorem 2.4.1 that is representation of strict Fréchet–Hilbert spaces in the form of a strict projective limit of the sequence of its complemented subspaces and the representation of its strongly conjugate space in the form of a strict inductive limit of the same sequence of its complemented subspaces.

It should also be noted that if E is a strict Fréchet–Hilbert space, then, by Theorem 2.4.1, the canonical map K_n is the projector of E onto H_n, and its restriction K_{n,H_n} to H_n is a topological isomorphism of H_n onto $E/\operatorname{Ker} p_n$ for every $n \in \mathbb{N}$. Therefore, each element $x \in \mathbb{N}$ is identified with the sequence $\{k_n x\}$, and since $E/\operatorname{Ker} p_n$ is isomorphic to the subspace H_n, then the element $k_n x$ is also identified with the element $h_n = \overset{(n)}{x} \in H_n$. The element $\overset{(n)}{x}$ is called the trace of x in H_n. If, in particular, $h \in H_n$, then its trace in H_n coincides with h, i.e. $k_n h = h = \overset{(n)}{h}$. Moreover, for any $h_1, h_2 \in H_n$, the following equalities are true:

$$(h_1, h_2)_n = (h_1, h_2)_{n,H_n} = \langle k_n h_1, k_n h_2 \rangle_n. \tag{4.2.26}$$

For example, in the case of the space $\omega = C^N$, the subspace H_n, mentioned in Theorem 2.4.1, is identified with C^n and for $x = \{x_n\} \in \omega$, $k_n(x) = \overset{(n)}{x} = (x_1, \dots, x_n, 0, \dots)$ is a trace of element x in H_n.

In the case of the space $L^2_{loc}(\Omega)$, where $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$, $\Omega_n \subset \operatorname{int} \Omega_{n+1}$ $(n \in \mathbb{N})$, the subspace H_n mentioned in Theorem 2.4.1 is identified with $L^2_0(\Omega_n)$, the space of functions summable in a square on Ω_n and equal to zero outside Ω_n. The trace of the function $f \in L^2_{loc}(\Omega_n)$ in H_n is its restriction to Ω_n.

Using Theorem 2.4.1, one can easily prove that for each element x of strict Fréchet–Hilbert space E, the sequence of its traces $\{\overset{(n)}{x}\}$ converges to x in E.

Therefore, the dual space $E' = \bigcup_{n\in\mathbb{N}} H_n$ to the strict Fréchet–Hilbert space E is everywhere dense in E, i.e. $E = \overline{\bigcup_{n\in\mathbb{N}} H_n}^{E}$.

Now let $(E, \mathfrak{T}) = s \cdot \lim\limits_{\leftarrow}(H_n, p_{n,H_n})$ be a strict Fréchet–Hilbert space and L be a symmetric operator on E with dense domain $D(L)$. Let $L^{(n)}$ denote its restriction to H_n. If $D(L) \cap H_n$ is dense everywhere in H_n and leaves each H_n invariant, i.e. $L(H_n) \subset H_n$, then $L^{(n)}$ is symmetric in H_n. According to ([139], p. 252, Lemma 2), every symmetric operator on a separable Hilbert space has a sequence of invariant subspaces. Let L_n denote the projection of L on the quotient space $E/\operatorname{Ker} p_n$, defined by the equality

$$L_n(K_n x) = K_n(Lx). \tag{4.2.27}$$

By virtue of the above, L defines a symmetric operator on H_n as well.

Theorem 4.2.7. *Let $(E, \mathfrak{T}) = s\cdot\lim\limits_{\leftarrow}(H_n, p_{n,H_n})$ be a strict Fréchet–Hilbert space, $L : E \to E$ be a linear operator with dense domain $D(L)$ and $L(H_n) \subset H_n$ for each $n \in \mathbb{N}$. Then the following statements are true:*

a) *L is a symmetric operator in E if and only if the operators L_n are symmetric in $(E/\widetilde{\operatorname{Ker} p_n}, \widehat{p}_n)$ for each $n \in \mathbb{N}$ with dense domain $D(L_n) = K_n(D(L))$.*

b) *For $h \in D(L) \cap H_n$, the following equalities are true:*

$$L_n(K_n h) = Lh = L^{(n)}h, \tag{4.2.28}$$

where L_n is defined by the equality (4.2.27) *and the operator L is defined on $D(L)$ by the equality*

$$L\phi = L(\{\varphi_n\}) = \{L_n\phi\} = \{L^{(n)}\phi_n\}, \tag{4.2.29}$$

where $\{\phi_n\} = \{k_n\phi\}$.

c) *For the operator L to be symmetric, it is necessary and, if $K_n(D(L)) \subset D(L^{(n)}) = D(L) \cap H_n$, it is sufficient that $L^{(n)}$ were symmetric in H_n for each $n \in \mathbb{N}$.*

d) *If for each $n \in \mathbb{N}$ the equality $D(L_n) = H_n$ holds, then L is a continuous self-adjoint operator if and only if $D(L) = E$.*

Proof. a) Let L be a symmetric operator on E and $h_1, h_2 \in D(L_n)$. Since $D(L_n) = K_n(D(L))$, then there exist $x, y \in D(L)$ such that $k_n x = h_1$ and $k_n y = h_2$. Then for these x, y and $n \in \mathbb{N}$ we have

$$(Lx, y)_n = (x, Ly)_n. \tag{4.2.30}$$

By the definition of the inner product on the quotient space $E/\operatorname{Ker} p_n$ we obtain the equality

$$\langle k_n Lx, k_n y\rangle_n = \langle k_n x, k_n Ly\rangle_n.$$

Further, by virtue of (4.2.27), we have

$$\langle L_n k_n x, k_n y\rangle_n = \langle k_n x, L_n k_n y\rangle_n. \tag{4.2.31}$$

But from here, by definition, for h_1 and h_2 the following equality is true:

$$\langle L_n h_1, h_2\rangle_n = \langle h_1, L_n h_2\rangle_n.$$

Now let $x, y \in D(L)$ and $n \in \mathbb{N}$, then if $k_n x = h_1$ and $k_n y = h_2$, then $h_1, h_2 \in k_n(D(L)) = D(L_n)$. Due to the symmetry of L_n, we obtain that the equality (4.2.31) is true. Hence, repeating the above reasoning in reverse order, we obtain that (4.2.30) is true, i.e. L is a symmetric operator on E.

b) As noted above, in the case of the strict Fréchet–Hilbert space E, the canonical map $k_n : E \to E/\operatorname{Ker} p_n = H_n$ is the projector of E onto H_n, and its restriction to H_n is an isomorphism of H_n onto $(E/\operatorname{Ker} p_n, \widehat{p}_n)$. Therefore, if $h \in H_n$, then $k_n h = h = \overset{(n)}{h}$. From here and from the condition we see that (4.2.28) is true.

Let $\phi \in D(L)$. Since the space E is represented as the strictly projective limit of the sequence of its complemented Hilbert spaces $\{(H_n, p_{n,H_n})\}$ with respect to the mappings π_{nm} $(n \le m)$, we obtain that $\phi = \{\phi_n\} = \{k_n\phi\}$, where $k_n\phi \in k_n(D(L))$ and $\pi_{nm}k_m\phi = k_n\phi$ $(n \le m)$. Next, we have

$$\begin{aligned}\pi_{nm}(L_m\phi_m) &= \pi_{nm}L_m k_m\phi = \pi_{nm}k_m(L\phi) = k_n(L\phi)\\ &= L_n k_n\phi = L_n\phi_n \ \ (n \le m).\end{aligned}$$

Therefore, the sequence $\{L_n\phi_n\}$ defines an element $L\phi$ of the space E. By virtue of the first part of statement b), we also obtain that

$$L\phi = \{L^{(n)}\phi_n\}.$$

Let L be a symmetric operator on E. Then for any $n \in \mathbb{N}$ and $h_1, h_2 \in D(L) \cap H_n$ we have that

$$(Lh_1, h_2)_n = (h_1, Lh_2)_n.$$

Since $L(H_n) \subset H_n$, by virtue of (4.2.26) it follows that

$$(L^{(n)}h_1, h_2)_{n,H_n} = (h_1, L^{(n)}h_2)_{n,H_n}. \tag{4.2.32}$$

It is easy to verify that $D(L) \cap H_n$ is everywhere dense in H_n. Therefore, by hypothesis, $L^{(n)}$ is symmetric in H_n.

Let now $L^{(n)}$ be symmetric for every $n \in \mathbb{N}$, i.e. (4.2.32) is valid for any $h_1, h_2 \in D(L) \cap H_n$.

Let $x, y \in D(L)$, then $h_1 = k_n x$ and $h_2 = k_n y$ belong to $D(L^{(n)})$. Therefore, (4.2.32) is true for them, and in (4.2.26) and (4.2.28) we also have that for each $n \in \mathbb{N}$ the equality is true:

$$\langle L_n k_n h_1, k_n h_2 \rangle_n = \langle k_n h_1, L_n k_n h_2 \rangle_n,$$

i.e. L_n is symmetric for every $n \in \mathbb{N}$. According to statement a) we obtain that L is symmetric in E.

d) Let $D(L_n) = H_n$ for each $n \in \mathbb{N}$ and L be a self-adjoint operator, then according to (4.2.29) $L\phi$ exists for any $\phi = \{k_n \phi\} \in E$ and, due to the self-adjointness of L, $\phi \in D(L)$. Conversely, if $D(L) = E$, then, by virtue of the generalized Hellinger–Toeplitz theorem, we obtain that L is self-adjoint, continuous, and belongs to the class $L_0(E)$. □

Now let $(H, \|\cdot\|)$ be a Hilbert space, $\{H_n\}$ be an increasing sequence of its closed subspaces such that $H = \overline{\bigcup_{n \in \mathbb{N}} H_n}^H$. Let $j_{nm} : H_n \to H_m$ and $j_n : H_n \to \bigcup_{n \in \mathbb{N}} H_n = F$ be sequences of isometric embeddings for which the equalities $j_m \cdot j_{nm} = j_n$ $(n \leq m)$ hold, L be a symmetric operator in H with dense domain $D(L)$ and $L^{(n)}$ be a restriction of L on H_n with domain $D(L^{(n)}) = D(L) \cap H_n$ or, more precisely, $D(L) = \bigcup_{n \in \mathbb{N}} j_n(D(L^{(n)}))$ (we sometimes omit j_n and j_{nm} if this does not lead to misunderstanding). In the inductive limit topology $\mathfrak{T}$, the space F is a strict (LH)-space, i.e. a strict inductive limit of the sequence of Hilbert spaces $\{H_n\}$. By virtue of Theorem 2.4.1, its strong dual space $E = (F', \beta(F', F)) = s \cdot \lim\limits_{\leftarrow} H_n$ is a strict Fréchet–Hilbert space, i.e. the strict projective limit of the sequence $\{H_n\}$ with respect to the adjoint mappings $j'_{nm} : H'_m \to H'_n$ and $j'_{nm} : H_m \to H_n$ $(n \leq m)$. In these notations we have

Theorem 4.2.8. *Let $\{H_n\}$ be an increasing sequence of subspaces of the Hilbert space $(H, \|\cdot\|)$ such that $\overline{\bigcup_{n \in \mathbb{N}} H_n}^H = H$, let L be a symmetric operator in H, $L(H_n) \subset H_n$ and $L^{(n)}$ be the restriction of L to H_n. If for any $n \leq m$ the conditions $j'_{nm} L^{(m)} \phi_m = L^{(n)} j'_{nm} \phi_m$ are satisfied, then the equality*

$$\widetilde{L}(\{\phi_n\}) = \{L^{(n)} \phi_n\} \tag{4.2.33}$$

defines a symmetric operator on the strict Fréchet–Hilbert space $E = (F', \beta(F', F)) = s \cdot \lim\limits_{\leftarrow} H_n$, the strict projective limit of the sequence of its subspaces $\{H_n\}$, with respect to the mappings j'_{nm} $(n \leq m)$.

Proof. Indeed, by virtue of the above, the conjugate to the strict (LH)-space $F = s \cdot \lim\limits_{\rightarrow} H_n$ in the strong topology is the strict Fréchet–Hilbert space $E = s \cdot \lim\limits_{\leftarrow} H_n$ that is a strict projective limit of the sequence of spaces $\{H_n\}$ with respect to the mappings j'_{nm}. Therefore, each element of $\phi \in E$ is represented in the form $\phi = \{\phi_n\}$, $\phi_n \in H_n$, where $j'_{nm}\phi_m = \phi_n$ $(n \leq m)$ and $j'_n\phi = \phi_n$, i.e. $j'_m \cdot j'_{nm} = j'_n$ $(n \leq m)$. Let's check that the right-hand side in (4.2.33) is an element from E. In fact, we have that $j'_{nm}L^{(m)}\phi_m = L^{(n)}j'_{nm} \cdot \phi_m = L^{(n)}j'_{nm} \cdot j_m\phi = L^{(n)}j'_n \cdot \phi = L^{(n)}\phi_n$ $(n \leq m)$.

It is also obvious that the restriction of $\widetilde{L}$ to H_n coincides with $L^{(n)}$ and, by virtue of statement b) of Theorem 4.2.7, we obtain that $\widetilde{L}_n = L^{(n)}$. $\square$

Example. Let $H = L^2_0(\mathbb{R})$, $H_n = L^2_0[-n,n]$, $j_{nm} := L^2_0[-n,n] \to L^2_0[-m,m]$ be identical embedding, $j_n : L^2_0[-n,n] \to \bigcup\limits_{n\in\mathbb{N}} L^2_0[-n,n] = F$ and $Lx(t) = tx(t)$ be a well-known position operator in quantum mechanics. Obviously, $L(H_n) \subset H_n$ for every $n \in \mathbb{N}$. The mappings j'_{nm} and j'_n are operators of restriction of functions to $[-n,n]$. The equalities $j'_{nm}L^{(m)}\phi_m = L^{(n)}j'_{nm}\phi_m$ indicate the commutativity of the two operations of restriction and multiplication on the argument t, i.e. $j'_{nm}L^{(m)}\phi_m = j'_{nm}t\phi_m = t\phi_n$ and $L^{(n)}j'_{nm}\phi_m = L^{(m)}\phi_n$. Then, by virtue of the equalities

$$\widetilde{L}\phi(t) = \{L^{(n)}\phi_n\} = t\phi(t), \quad \phi \in L^2_{loc}(\mathbb{R}),$$

the continuation of the symmetric operator L from the Hilbert space $L^2(\mathbb{R})$ to the strict Fréchet–Hilbert space $L^2_{loc}(\mathbb{R})$ is defined. Moreover, $D(\widetilde{L}_n) = D(L^{(n)}) = L^2_0[-n,n]$ and therefore, by virtue of statement d) of Theorem 4.2.7, $D(\widetilde{L}) = L^2_{loc}(\mathbb{R})$, $\widetilde{L}$ is a continuous and self-adjoint operator and $\widetilde{L} \in L_0(L^2_{loc}(\mathbb{R}))$.

Let us now indicate how Theorem 4.2.8 can be applied by the self-adjoint differential operator constructed in [98, 100]. Namely, in the notation considered above, for a sequence of subspaces $\{H_n\}$ of a Hilbert space $(H, \|\cdot\|)$ there is a sequence of self-adjoint operators L_n in the spaces H_n for which $D(L_n) \subset H_n$ and $j_{nm}D(L_n) \subset D(L_m)$ $(n \leq m)$. According to ([98], Theorem 1.1), if for any $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that for all $m > n \geq n_0(\varepsilon)$ and any $\phi_n \in D(L_n)$ the inequality

$$\|(L_m j_{nm} - j_{nm}L_n)\phi_n\| \leq \varepsilon\,(\|\phi_n\| + \|L_m j_{nm}\phi_n\| + \|L_n\phi_n\|)$$

holds, then through equality

$$L_\infty\phi = \lim_{m\to\infty} j_m L_m j_{nm}\phi_n$$

for $\phi = j_n\phi_n \in D_\infty = \bigcup_{n\in\mathbb{N}} j_n D_n$ we define an essentially linear self-adjoint operator L_∞ on D_∞.

Moreover, if H_n are invariant subspaces under the operator L_∞, $j'_{nm}L_\infty^{(m)}\phi_m = L_\infty^{(n)}j'_{nm}\phi_m$ $(n \le m)$, then by Theorem 4.2.8 the equality

$$\widetilde{L}_\infty(\{\phi_n\}) = \{L_\infty^{(n)}\phi_n\}$$

for $\{\phi_m\} \in E = (F', \beta(F', F)) = s \cdot \lim_{\leftarrow} H_n$ a symmetric operator is defined on the strict Fréchet–Hilbert space E, where $L_\infty^{(n)}$ – restriction of L_∞ to H_n.

4.3 Generalization of the Ritz method for operator equations in Fréchet–Hilbert spaces

4.3.1 Equation with symmetric and positive definite operators in Fréchet–Hilbert spaces

Let again E be a Fréchet space with a generating non-decreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$ and let $A : D(A) \subset E \to E$ be a symmetric and positive definite operator with dense domain $D(A)$ and the image $R(A)$. Then, by virtue of the corollary of Theorem 4.2.3, A has a unique self-adjoint extension $\widetilde{A}$ such that the equation

$$\widetilde{A}u = f \tag{4.3.1}$$

has a unique solution u_0 for each $f \in E$.

It may turn out that $u_0 \in D(A)$, then u_0 will be the classical solution of the equation

$$Au = f. \tag{4.3.2}$$

If $u_0 \notin D(A)$, then we call it a generalized solution of the equation (4.3.2).

The solution of the equation (4.3.1) belongs to the energetic space E_A defined in Section 4.2.1. Indeed, the equation (4.3.1) has a solution $u_0 \in E$ if and only if the equation

$$\widetilde{A}_n k_n u = k_n f$$

has a solution $k_n u_0$ for each $n \in \mathbb{N}$. It is well known that $k_n u_0$ belongs to the energetic space $H_{\widetilde{A}_n}$ of the operator $\widetilde{A}_n$. On the other hand, consider the equation

$$A_n k_n u = k_n f \tag{4.3.3}$$

in Hilbert space $E_n = (\widetilde{E/\ker\|\cdot\|_n}, \widehat{\|\cdot\|_n})$. By Lemma 4.2.1, the operator A_n is symmetric and positive definite and, therefore, according to the Ritz method in energetic spaces ([**?**], p. 26), the equation (4.3.3) has a generalized solution $\widehat{u}_n$ belonging to the energetic space $(H_{A_n}, [\quad]_{A_n})$. $\widehat{u}_n$ is a solution to the equation

$$\widetilde{A}_n u_n = k_n f,$$

where $\widetilde{A}_n$ is a self-adjoint extension of the operator A_n, which provides a minimum to the energetic functional

$$F_n(k_n u) = \langle A_n k_n u, k_n u\rangle_n - \langle k_n f, k_n u\rangle_n - \langle k_n u, k_n f\rangle_n. \tag{4.3.4}$$

By virtue of Theorem 4.7.3 ([20], p. 79), the energetic space of a positive definite operator and its Friedrich's extensions coincide and therefore $(H_{A_n}, [\,\cdot\,]_{A_n}) = (H_{\widetilde{A}_n}, [\,\cdot\,]_{\widetilde{A}_n})$. Since equations with operators A_n and $\widetilde{A}_n$ have unique solutions, $\widehat{u}_n = k_n u_0 \in H_{A_n} \subset E_n$. From this we can conclude that the solution u_0 also belongs to the energetic space E_A of the operator A, which is the projective limit of the sequence of spaces $\{(H_{A_n}, [\,\cdot\,]_{A_n})\}$.

This section generalizes the Ritz method for approximately solving the equation (4.3.1).

Let us first note that the above-mentioned operator A can be represented as follows: $Ax = \{k_n Ax\} = \{A_n k_n x\}$. From the positive definiteness of the operators A and A_n it follows that there are inverse operators A^{-1} and A_n^{-1}, which are related by the equalities

$$A^{-1}x = \{k_n A^{-1}x\} = \{(A^{-1})_n k_n x\},$$

where $(A^{-1})_n$ is the projection of the inverse operator A. Obviously, for each $n \in \mathbb{N}$, the equalities $A_n^{-1} = (A^{-1})_n$ are valid. The operators A^{-1} and A_n^{-1} are also related by the equalities

$$\pi_{nm} A_m^{-1} k_m x = \pi_{nm} k_m A^{-1} x = k_n A^{-1} x = A_n^{-1}(k_n x) \quad (n \leq m).$$

Similar representations are also valid for the operators $\widetilde{A}$ and $\widetilde{A}^{-1}$.

It should be noted that if E is a Fréchet–Hilbert space with a nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, then for a fixed $f \in E$ the inner products $(f, u)_n$ generate continuous functionals on E, which, generally speaking, differ from each other for different $n \in \mathbb{N}$. The equality

$$(f, u)_n = \langle k_n f, k_n u\rangle_n \tag{4.3.5}$$

defines a linear continuous functional on the Hilbert space $E_n = (E/\widetilde{\operatorname{Ker}\|\cdot\|_n}, \widehat{\|\cdot\|_n})$ $(n \in \mathbb{N})$. From the inequalities

$$|(x, u)|_n = |\langle k_n f, k_n u\rangle| \leq \|\widehat{f}\|_n \cdot \|\widehat{u}\|_n \leq \|\widehat{f}\|_n \gamma_n^{-1} [u]_{A_n}$$

it follows that the indicated functionals define continuous linear functionals on H_{A_n} $(n \in \mathbb{N})$. Therefore, there are the elements $u_n \in H_{A_n}$ satisfying the equalities

$$\langle k_n f, k_n u\rangle_n = [u_n, k_n u]_{A_n}, \tag{4.3.6}$$

where $[\cdot,\cdot]_{A_n}$ is the inner product of H_{A_n} $(n \in \mathbb{N})$ defined by (4.2.6).

So, each $f \in E$ generates a sequence of elements $\{u_n\}$, where $u_n \in H_{A_n}$ (in what follows we will consider the right side of the equation (4.3.1) as f). In the case when $k_n f \in R(A_n)$, i.e. when $k_n f$ belongs to the image of the operator A_n, we have

$$\langle k_n f, k_n u\rangle_n = \langle A_n A_n^{-1} k_n f, k_n u\rangle_n = [A_n^{-1} k_n f, k_n u]_{A_n} = [u_n, k_n u]_{A_n}.$$

It follows that $u_n = A_n^{-1}(k_n f)$, i.e. u_n is a solution to the equation (4.3.3). In the case when there is a solution $u_0 = \{\overline{u}_n\} \in E$ to the equation (4.3.1), i.e. $\overline{u}_n$ are the minimum points of the functionals (4.3.4), then the functionals (4.3.5) are generated by the elements of $\overline{u}_n \in H_{A_n}$.

4.3.2 Definition of an approximate solution and convergence of its sequence

Let $A : E \to E$ be the symmetric and positive definite operator of the Fréchet space E into itself and E_A be the energetic space of the operator A. The topology $\mathfrak{T}_A$ of the space E_A, generated by the non-decreasing sequence $\{[\cdot]_n\}$, will be metrized using the metric (2.5.12), i.e. for $x, y \in E_A$ we set

$$d(x,y) = |x-y| \\ = \begin{cases} [x-y]_1, & \text{when } [x-y]_1 \geq 1, \\ 2^{-n+1}, & \text{when } [x-y]_n \leq 2^{-n+1} \\ & \qquad \text{and } [x-y]_{n+1} \geq 2^{-n+1} \ (n \in \mathbb{N}), \\ [x-y]_{n+1}, & \text{when } 2^{-n} \leq [x-y]_{n+1} < 2^{-n+1} \ (n \in \mathbb{N}), \\ 0, & \text{when } x-y=0. \end{cases} \tag{4.3.7}$$

For the balls $K_r = \{x \in E_A;\ d(x,0) \leq r\}$ $(r \in R^+)$ of this metric the relations $K_r = rV_n$ are valid, where $V_n = \{x \in E_A;\ [x]_n \leq 1\}$, when

$$r \in I_n = \begin{cases} [1, \infty[& \text{for } n = 1, \\ [2^{-n+1}, 2^{-n+2}[& \text{for } n \geq 2. \end{cases} \tag{4.3.8}$$

The Minkowski functionals $q_r(\cdot)$ of the balls K_r have the following form:

$$q_r(x) = r^{-1}[x]_n \quad \text{at } r \in I_n. \tag{4.3.9}$$

It should be noted that if the space E_A is a normed space with the energetic norm $[\cdot]$, with a unit ball S, and we assume that $V_1 = V_2 = \cdots = V_n = S$, then constructed for such case quasinorm (4.3.7) will coincide with the initial norm $[\cdot]$. Since in the considering case, the seminorms $[\,\cdot\,]_n$ are generated by semi-inner products, by virtue of (4.3.9), $q_r(\cdot)$ are also generated by the semi-inner products. Moreover, for $r \in I_n$ the seminorms $q_r(\cdot)$ and $[\,\cdot\,]_r$ differ from each other only by the positive factor r^{-1} (the Minkowski functionals K_r and S also differ in the normed space E) and therefore their unit balls are similar. Thus, our results contain the classical case of Hilbert spaces.

Let $\{\phi_j\} \subset E_A$ be a linearly independent sequence and G_m be a subspace of E_A spanned by $\phi_1, \ldots, \phi_m$. We call a sequence of subspaces $\{G_m\}$ extremely dense in E_A (similar to the terminology from [99]) if for any function $u \in E$ the sequence

$$d(u, G_m) = \inf\{d(u, g);\ g \in G_m\} \to 0 \quad \text{as} \ \ m \to \infty.$$

Proposition 4.3.1. *In the above notation, the following statements are equivalent:*

a) *The sequence of subspaces $\{G_m\}$ is extremely dense in E_A.*

b) *For each $l \in \mathbb{N}$ and any function $u \in E_A$, a sequence of numbers*

$$\inf\{[u-g]_l;\ g \in G_m\} \to 0 \quad \textit{as} \ \ m \to \infty.$$

c) *The closure of the set $\bigcup\limits_{m\in\mathbb{N}} G_m$ in E_A coincides with E_A.*

Lemma 4.3.2. *Let E be a Fréchet space with a nondecreasing sequence of seminorms $\{[\,\cdot\,]_n\}$, with metric* (4.3.7), *and $G \subset E$ be its closed convex subset. Then, if for $u \in E$ the equality* $\inf\{[u-g]_n;\ g \in G\} = r$ *holds and $r \in I_n$, then $d(u, G_m) = r$. If, in addition, $r \in \operatorname{int} I_n$ ($n \in \mathbb{N}$), then the converse is also true.*

The proof of Lemma 4.3.2 for the metric (4.3.7) does not differ significantly from the proof of Proposition 3.1.2 and we omit it.

Proof of Proposition 4.3.1. a) $\Rightarrow$ b) Let $u \in E_A$, $d(u, G_m) \to 0$ for $m \to \infty$ and $l \in \mathbb{N}$. Then there exists $m_0 = m_0(l) \in \mathbb{N}$ such that for $m \geq m_0$ the inequality $d(u, G_m) < \sup I_l$ holds. If $d(u, G_m) \in I_l$, then, by Lemma 4.3.2,

$$\inf\{[u-g]_l;\ g \in G_{m_0}\} \leq d(u, G_{m_0}) < \sup I_l.$$

That's why

$$\inf\{[u-g]_l;\ g \in G_{m_0}\} < \sup I_l, \ \ \text{when} \ \ m \geq m_0.$$

If $d(u, G_{m_0}) \in I_{l+p}$ for some $p \in \mathbb{N}$, then, again by Lemma 4.3.2, we obtain

$$\begin{aligned}\inf\{[u-g]_l;\ g \in G_{m_0}\} &\le \cdots \le \inf\{[u-g]_{l+p};\ g \in G_{m_0}\} \\ &\le d(u, G_{m_0}) < \sup I_{l+p} < \sup I_l.\end{aligned}$$

Therefore, the condition of statement b) is satisfied.

b) $\Rightarrow$ a) It is necessary to prove that for every $u \in E_A$ and $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that for all $m \ge m_0$ the inequality $d(u, G_m) < \varepsilon$ holds. Let us assume the opposite. Then there exist $u_0 \in E_A$ and $\varepsilon_0 > 0$ such that $\lim\limits_{m\to\infty} d(u, G_m) = \alpha \ge \varepsilon_0$. Let first $\alpha \in I_l$, i.e. $\alpha \ge 2^{-l+1}$, so there exists $m_1 = m_1(\varepsilon)$ such that $d(u_0, G_m) \in I_l$ for $m \ge m_1$. If $\alpha \in \operatorname{int} I_l$, then by Lemma 4.3.2 and from the strong proximality of finite-dimensional subspaces in Fréchet spaces E with respect to the metric (4.3.7) there exist $g_m \in G_m$ such that

$$d(u_0, G_m) = d(u_0, g_m) = [u_0 - g_m]_l = \inf\{[u_0 - g]_l;\ g \in G_m\} \to \alpha \ \text{ as } \ m \to \infty.$$

Since $\alpha > 0$, this contradicts our assumption.

If $\alpha = 2^{-l+1}$, then either $d(u_0, G_m) \in \operatorname{int} I_l$ and

$$d(u_0, G_m) = [u_0 - g_m]_l = \inf\{[u_0 - g]_l;\ g \in G_m\} \to \alpha \quad \text{as } \ m \to \infty,$$

which again leads to a contradiction, or starting from some $m_1 \in \mathbb{N}$, the equalities $d(u_0, G_m) = \alpha = d(u_0, g_m)$ for $m \ge m_1$ are valid. This means that $(u_0 + G_m) \cap K_{2^{-l+1}} = (u_0 + G_m) \cap V_l \ni u_0 + g_m$ and $(u + G_m) \cap \operatorname{int} K_{2^{-l+1}} = (u_0 + G_m) \cap V_l = \emptyset$. However, according to the condition, we have

$$[u_0 + g'_m]_{l+1} = \inf\{[u_0 + g]_{l+1};\ g \in G_m\} \to 0 \quad \text{as } \ m \to \infty.$$

Therefore, for V_{l+1} there is a number $m_2 \in \mathbb{N}$ such that

$$u_0 + g'_m \in V_{l+1} = K_{2^{-l}} \subset \operatorname{int} V_l \quad \text{as } \ m \ge m_1,$$

and this contradicts the condition $(u_0 + G_m) \cap V_l = \emptyset$ for $m \ge \max(m_1, m_2)$.

The proof of the equivalence of a) $\Leftrightarrow$ c) is not difficult and we omit it. □

Theorem 4.3.3. *Let E be a Fréchet space with a non-decreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, A be a symmetric and positive definite operator, E_A be the energetic space of the operator A with a non-decreasing sequence of hilbertian seminorms $\{[\,\cdot\,]_n\}$, with the metric* (4.3.7)*, $\{\phi_k\}$ be a sequence of basis functions from $D(A) \subset E_A$, $G_m \subset E_A$ be the subspace spanned by the functions $\phi_1, \dots, \phi_m$, $\operatorname{Ker}[\,\cdot\,]_n \cap G_m = \{0\}$ and let $u_0 = \{u_0^n\} \subset E_A$ be a generalized solution of the equation* (4.3.2)*. Then the following statements are equivalent:*

a) $\quad d(u_0, G_m) = d(u_0, u_m) = r \in \operatorname{int} I_l;$ (4.3.10)

b) $\inf\{[u_0 - g]_l;\ g \in G_m\} = [u_0 - u_m]_l = r \in \operatorname{int} I_l;$ (4.3.11)

c) $u_m = \sum\limits_{j=1}^{m} a_j \phi_j$, *where the coefficients* $a_1, \ldots, a_m$ *satisfy the system of equations*

$$\sum_{k=1}^{m} [\phi_k, \phi_j]_l a_k = (f, \phi_j)_l, \quad j = 1, \ldots, m, \tag{4.3.12}$$

and

$$r = \sqrt{\frac{G(u_0, \phi_1, \ldots, \phi_m)_l}{G(\phi_1, \ldots, \phi_m)_l}} \in \operatorname{int} I_l, \tag{4.3.13}$$

where $G(\phi_1, \ldots, \phi_m)_l = \det([\phi_k, \phi_j]_l)$ *is the Gram determinant.*

Proof. a) $\Rightarrow$ b) follows from Lemma 4.3.2, taking into account the fact that the finite-dimensional subspace G_m is proximal in E_A with respect to the metric (4.3.7) and with respect to the seminorm $[\,\cdot\,]_l$, i.e. the infimums in (4.3.10) and (4.3.11) are achieved on the same element $u_m \in G_m$ for $r \in \operatorname{int} I_l$.

b) $\Rightarrow$ c) Statement b) means that $u_m = \sum\limits_{j=1}^{m} a_j \phi_j \in G_m$ is the best approximation for u_0 in G_m with respect to the hilbertian seminorm $[\,\cdot\,]_l$. This is equivalent to saying that $k_{A,l} u_m$ is the best approximation of $k_{A,l} u_0$ with respect to the norm $\widehat{[\,\cdot\,]}_1$ of the Hilbert space $E_{A,l}$, i.e.

$$\big[k_{A,l}\widehat{u_0 - k}_{A,l} u_m\big]_l = \inf\big\{\big[k_{A,l}\widehat{u_0 - k}_{A,l} g\big]_l;\ g \in G\big\} = r \in \operatorname{int} I_l, \tag{4.3.14}$$

where $k_{A,l} : E_A \to E_A/\operatorname{Ker}[\,\cdot\,]_l$ is the canonical mapping, and $\widehat{[\,\cdot\,]}_l$ is the norm on $E_A/\operatorname{Ker}[\,\cdot\,]_l$, associated with the seminorm $[\,\cdot\,]_l$. Since $k_{A,l} g = \sum\limits_{j=1}^{m} a_j k_{A,l} \phi_j$, (4.3.14) can be written in the form

$$\begin{aligned} &\big[k_{A,l}\widehat{u_0 - k}_{A,l} u_m\big]_l \\ &= \inf\Big\{\Big[k_{A,l}\widehat{u_0 - \sum_{j=1}^{m}} a_j k_{A,l} \phi_j\Big]_l;\ a_1, \ldots, a_m \in R\Big\} = r \in \operatorname{int} I_l. \end{aligned} \tag{4.3.15}$$

According to ([1], p. 25), (4.3.15) is satisfied if and only if the coefficients $a_1, \ldots, a_m$ satisfy the system of equations

$$\sum_{i=1}^{m} \big[k_{A,l}\widehat{\phi_i, k}_{A,l}\phi_j\big]_l a_i = \big[k_{A,l}\widehat{u_0, k}_{A,l}\phi_j\big]_l, \quad j = 1, \ldots, m,$$

and

$$\left[k_{A,l}\widehat{u_0 - k}_{A,l}u_m\right]_l = \sqrt{\frac{G(k_{A,l}u_0, k_{A,l}\widehat{\phi_1, \ldots}, k_{A,l}\phi_m)_l}{G(k_{A,l}\phi_1 \widehat{, \ldots, k}_{A,l}\phi_m)_l}} = r \in \operatorname{int} I_l,$$

where

$$G(k_{A,l}u_0, \ldots, k_{A,l}\phi_m)_l = \begin{vmatrix} [k_{A,l}\widehat{u_0, k}_{A,l}u_0]_l, & \ldots, & [k_{A,l}\widehat{\phi_m, k}_{A,l}u_0]_l \\ [k_{A,l}\widehat{u_0, k}_{A,l}\phi_1]_l, & \ldots, & [k_{A,l}\widehat{\phi_m, k}_{A,l}\phi_1]_l \\ \cdots & \cdots & \cdots \\ [k_{A,l}\widehat{u_0, k}_{A,l}\phi_m]_l, & \ldots, & [k_{A,l}\widehat{\phi_m, k}_{A,l}\phi_m]_l \end{vmatrix}$$

and

$$G(k_{A,l}\phi_1 \widehat{, \ldots, k}_{A,l}\phi_m)_l = \begin{vmatrix} [k_{A,l}\widehat{\phi_1, k}_{A,l}\phi_1]_l, & \ldots, & [k_{A,l}\widehat{\phi_m, k}_{A,l}\phi_1]_l \\ \cdots & \cdots & \cdots \\ [k_{A,l}\widehat{\phi_1, k}_{A,l}\phi_m]_l, & \ldots, & [k_{A,l}\widehat{\phi_m, k}_{A,l}\phi_m]_l \end{vmatrix}.$$

By the definition of thhe inner products $[\widehat{\cdot}]_1$, we have the equalities

$$G(k_{A,l}u_0, k_{A,l}\widehat{\phi_1, \ldots}, k_{A,l}\phi_m)_l = G(u_0, \phi_1, \ldots, \phi_m)_l$$

and

$$G(k_{A,l}\phi_1 \widehat{, \ldots, k}_{A,l}\phi_m)_l = G(\phi_1, \ldots, \phi_m)_l.$$

As is known ([1], p. 17), $G(k_{A,l}\phi_1, \ldots, k_{A,l}\phi_m) \neq 0$ only if the vectors $k_{A,l}\phi_1, \ldots, k_{A,l}\phi_m$ are linearly independent. For this, it is enough that the restriction of $k_{A,l}$ to G_m is injective, i.e. so that $\operatorname{Ker} k_{A,l} \cap G_m = \{0\}$. Since $\operatorname{Ker} k_{A,1} \supset \operatorname{Ker} k_{A,2} \supset \cdots$, it is sufficient to require that $\operatorname{Ker} k_{A,1} \cap G_m = \{0\}$. This is equivalent to $\operatorname{Ker}[\,\cdot\,]_1 \cap G_m = \{0\}$. In particular, this condition is satisfied if $[\,\cdot\,]_1$ is a norm on E_A. We must take this circumstance into account when choosing basis functions. These arguments prove the validity of conditions (4.3.12) and (4.3.13), taking into account the equalities

$$\left[k_{A,l}\widehat{u_0, k}_{A,l}\phi_j\right]_l = [u_0, \phi_j]_l = [\widetilde{A}^{-1}f, \phi_j]_l = (\widetilde{A}A^{-1}f, \phi_j)_l = (f, \phi_j)_l,$$

where $\widetilde{A}$ is a continuation of A that exists due to the corollary of Theorem 4.2.3.

c) $\Rightarrow$ b) is proved by carrying out the reverse order of reasoning that was carried out in proving the implication b) $\Rightarrow$ c). □

It should be noted that the requirement $r \in \operatorname{int} I_1$ is essential only for proving the implication a) $\Rightarrow$ b).

It should also be noted that, under the conditions of Theorem 4.3.3, the generalized solution u_0 provides a minimum on E_A to the functional

$$F(u) = |u - u_0|^2 - |u_0|^2,$$

where $|\cdot|$ is a quasinorm of the metric (4.3.7), and u_m provides a minimum to this functional on G_m.

Theorem 4.3.3 suggests that an approximate solution should be sought in the form $u_m = \sum\limits_{k=1}^{m} a_k \phi_k$, where the coefficients $a_1, \ldots, a_m$ are determined from the system (4.3.12). In this case, we will say that the approximate solution was constructed using the Ritz method. If it is additionally known that the solution is found with an accuracy of $\varepsilon > 0$, then we begin solving the system (4.3.12) with respect to the l-th semi-inner product, for $\varepsilon \in I_l$, where the intervals I_l were defined by equalities (4.3.8). In this case, we also obtain the best approximation of the generalized solution u_0 in G_m with respect to the semi-inner product $[\cdot]_l$, i.e. the best approximation of the l-th projection $k_{l,A}u_0$ with respect to the inner product $\widehat{[\cdot]}_l$ of the space $E_{A,l}$. From the extremely density of the sequence $\{\phi_j\}$ and by virtue of Proposition 4.3.1 we obtain that

$$[u_0 - u_m]_l = [k_{A,1}u_0 - k_{A,1}u_m]_l = \sqrt{\frac{G(u_0, \phi_1, \ldots, \phi_m)_l}{G(\phi_1, \ldots, \phi_m)_l}} \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore, there exists $m_0(l)$ such that for $m > m_0(l)$ the following inequalities are true:

$$[u_0 - u_m]_l < \sup I_l.$$

Theorem 4.3.4. *Let E be a Fréchet space with a nondecreasing sequence of hilbertian seminorms $\{\|\cdot\|_n\}$, A be a symmetric and positive definite operator, E_A be the energy space of operator A with an increasing sequence of hilbertian seminorms $\{[\cdot]_n\}$, with the metric* (4.3.7), *$\{\phi_k\}$ be a sequence of basis functions from $D(A) \subset E_A$, $G_m \subset E_A$ be a subspace spanned by the functions $\phi_1, \ldots, \phi_m$ and let $u_0 = \{u_0^n\} \in E_A$ be a generalized solution to the equation* (4.3.2). *Then there is a sequence of approximate solutions constructed by the Ritz method that converges to u_0, both in the energy space E_A and in the original space E. Moreover, for each $n \in \mathbb{N}$, there exists $k_0 = k_0(n)$ such that for each $k \geq k_0$ the following inequalities are true:*

$$[u_0 - u_k]_n \leq |u_0 - u_k|, \tag{4.3.16}$$

where $|\cdot|$ is a quasinorm of the metric (4.3.7).

Proof. As noted above, from the extremely density of the sequence $\{\phi_i\}$ and Proposition 4.3.1, we have that for an arbitrary $l_1 \in \mathbb{N}$ there exists $m_1(l_1)$ such that

$$[u_0 - u_{m_1(l_1)}]_{l_1} = [\widehat{k_{A,l_1}u_0 - k_{A,l_1}}u_{m_1(l_1)}]_{l_1} < \sup I_{l_1},$$

where $k_{A,l_1}u_0$ can be found as the limit of the sequence $\{k_{A,l_1}u_m\}$ constructed by solving the system (4.3.12), for $l = l_1$ by norm $\widehat{[\cdot]}_{l_1}$ for $m \to \infty$. Next, we choose $l_2 \geq (l_1 + 1, m_1(l_1))$, solve the system (4.3.12) again for $l = l_2$ and find $m_2(l_2)$ such that

$$[u_0 - u_{m_2(l_2)}]_{l_2} = [\widehat{k_{A,l_2}u_0 - k_{A,l_2}}u_{m_2(l_2)}]_{l_2} < \sup I_{l_2}.$$

Continuing this process, we find $m_k(l_k) \geq \max(l_{k-1} + 1, m_{k-1}(l_{k-1}))$ such that

$$[u_0 - u_{m_k(l_k)}]_{l_k} = [\widehat{k_{A,l_k}u_0 - k_{A,l_k}}u_{m_k(l_k)}]_{l_k} < \sup I_{l_k}.$$

It remains to prove that the sequence $\{u_{m_k(l_k)}\}$ converges to the solution u_0 in the space E_A and, consequently, in the space E. This follows from the following inequalities:

$$d(u_0, u_{m_k(l_k)}) < \sup I_{l_k} = 2^{-l_k+2} \quad \text{as} \quad l_k \geq 2,$$

since $u_0 - u_{m_k(l_k)}$ belongs to some ball of radius from the interval I_{l_k}. Therefore, in the space E_A, and therefore in E,

$$\lim_{k\to\infty} u_{m_k(l_k)} = u_0. \qquad \square$$

Remark 4.3.1. By virtue of the equalities $[\phi_k, \phi_j]_l = (A\phi_k, \phi_j)_l = \langle k_l A\phi_k, k_l\phi_j\rangle_l = \langle Ak_l\phi_k, k_l\phi_j\rangle_l = [k_l\phi_k, k_l\phi_j]_{A_l}$ and $(f, \phi_j)_l = \langle k_l f, k_l\phi_j\rangle_l$, the system (4.3.12) can be rewritten as

$$\sum_{i=1}^{m} [k_l\phi_k, k_l\phi_j]_{A_l} a_k = \langle k_l f, k_l\phi_j\rangle_l, \quad j = 1, \dots, m. \tag{4.3.17}$$

This system is the Ritz system for equations (4.3.3), which under the conditions of Theorem 4.3.4 has a generalized solution $k_l u_0$. If $\ker k_l \cap G_m = \{0\}$, i.e. the restriction of $\|\cdot\|_l$ to G_m is a norm, the system (4.3.17) has a unique solution and the sequence of approximate solutions converges with respect to the norm $[\cdot]_{A_l}$ to the solution $k_l u_0$ for $m \to \infty$. Due to the isometry of the spaces $(E_{A,l}, \widehat{[\cdot]}_l)$ and $(H_{A_l}, [\cdot]_{A_l})$, we find that $k_l u_0 = k_{A,l} u_0$.

4.4 Application of the Ritz method for the approximate solution of some operator equations. The space $D(A^\infty)$

Let H be a complex separable Hilbert space, let $A : D(A) \subset H \to H$ be a linear operator and $D(A)$ be dense in H. An element $u \neq 0$ of the energetic space H_A and a number λ are called *generalized eigenelement* and *generalized eigenvalue* of the operator A if they satisfy the identity

$$[u, \eta]_A = \lambda(u, \eta), \quad \forall \eta \in H_A.$$

The set of generalized eigenvalues is called *generalized spectrum*. It is known ([**?**], p. 92) that the generalized eigenvalues and eigenelements of a positive definite operator coincide with the ordinary eigenvalues and eigenelements of the Friedrichs extension of this operator.

The spectrum of an operator is called purely discrete ([111], p. 386) if it consists of a countable set of eigenvalues with a single limit point at infinity. If A is, in addition, a self-adjoint operator, then such a spectrum is called pure point spectrum ([160], p. 493).

By Rellich's theorem, a self-adjoint operator A has a purely point spectrum if and only if the embedding map of its domain $D(A)$ (with the norm $\|u\|_{D(A)} = |Au\| + \|u\|$) in H is compact ([160], p. 493).

If the generalized spectrum of a symmetric operator A is purely discrete and the sequence of corresponding generalized eigenelements is complete in H, then such a spectrum is called *discrete*.

Theorem 4.4.1 ([**?**], p. 98)**.** *Let a positive definite operator be such that any set bounded in the energy metric is compact in the metric of the original space. Then the generalized spectrum of this operator is discrete.*

The condition of this theorem can also be formulated as follows: the energy space is embedded in the original space completely continuously.

If A is a self-adjoint and positive definite operator, then the range $R(A)$ of the operator A coincides with H. It is clear that in this case the Friedrich's extension of A coincides with A and the generalized spectrum of the operator A coincides with its ordinary spectrum.

It is known ([111], p. 386) that if A is a self-adjoint positive definite operator, then its spectrum is purely discrete if and only if the energetic space of the operator A is embedded in the original space completely continuously. This proposal, together with Theorem 4.4.1, convinces us that

Theorem 4.4.2. *If the spectrum of a self-adjoint and positive definite operator A is purely discrete* (*i.e., is purely pointwise*)*, then this spectrum is discrete.*

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a self-adjoint operator with purely pointwise spectrum.

The topology of a well-known countable-Hilbert space

$$D(A^\infty) = \bigcap_{k=1}^{\infty} D(A^k)$$

can be generated with the sequences of hilbertian norms

$$\begin{gathered} \|x\|_n = (\|x\|^2 + \|Ax\|^2 + \cdots + \|A^n x\|^2)^{1/2}, \\ x \in D(A^\infty), \quad n \in \mathbb{N}_0 = N \cup \{0\}, \end{gathered} \tag{4.4.1}$$

which are generated by the inner products

$$\langle x, y \rangle_n = (x, y) + (Ax, Ay) + \cdots + (A^{n-1}x, A^n y), \quad x, y \in D(A^\infty).$$

The space $D(A^\infty)$ coincides with the space H if and only if the operator A is bounded. Note also that a sequence $\{h_k\}$ converges to h in the space $D(A^\infty)$ if and only if $A^n h_k$ converges to $A^n h$ in H for every $n \in \mathbb{N}_0$.

The spaces $D(A^\infty)$ were introduced by Mityagin [108] and were subsequently studied by Pitsch [126] and Triebel [159] for many differential operators. An important motive for introducing the spaces $D(A^\infty)$ was the question of the existence of a basis in special nuclear spaces.

In this section, the generalized Ritz method is used to approximate solution of the equation

$$Au = f \tag{4.4.2}$$

with positive definite operator A in the Hilbert space H in the case of a sufficiently smooth right-hand side.

4.4.1 Self-adjointness in Fréchet space $D(A^\infty)$ of restriction of self-adjoint operator A in Hilbert space H

It is well known that any Fréchet space is isomorphic to the subspace of product of sequences of Banach spaces. The space $D(A^\infty)$ is isomorphic to the subspace M of the Fréchet–Hilbert space H^N [201], the topology of which is generated by the sequence of semi-inner products

$$\langle x, y \rangle_n = (x_1, y_1) + \cdots + (x_n, y_n),\ x = (x_k),\ y = (y_k) \in H^N,\ n \in \mathbb{N}. \tag{4.4.3}$$

The indicated isomorphism (it is actually an isometry) is realized by the mapping

$$D(A^\infty) \ni x \to \mathrm{Orb}(A, x) := (x, Ax, \ldots, A^{n-1}x, \ldots) \in M \subset H^N.$$

This means that the space $D(A^\infty)$ is isomorphic to the space of all orbits $\mathrm{orb}\,(A, x)$ for the operator A at the point $x \in D(A^\infty)$, considered in the induced topology of the space H^N. Using this representation we define the operator $A^\infty : D(A^\infty) \to D(A^\infty)$ by the equality

$$A^\infty x = A^\infty(x, Ax, A^2x, \dots) = (Ax, A^2x, \dots),$$

i.e.

$$A^\infty(\mathrm{orb}(A, x)) = \mathrm{orb}(A, Ax). \tag{4.4.4}$$

In fact, A^∞ is the restriction to $D(A^\infty)$ of the operator A^N, defined on H^N, by the equality

$$A^N((x_k)) = (Ax_k) \in H^N.$$

We consider the equation (4.4.2) in the countable-Hilbert space $D(A^\infty)$ (the space $D(A^\infty)$ coincides with the space $C^\infty(A)$ ([140], p.345), the set of C^∞-elements of the operator A). Then it is proved the continuity and self-adjointness of A^∞ in the space $D(A^\infty)$. This notation for the space $D(A^\infty)$ acquires a new meaning that differs from the classical case where $D(A^\infty)$ was the whole symbol, where A^∞, if taken separately, meant nothing. From now on, the symbol $D(A^\infty)$ will also denote the definition domain of the operator A^∞ which obviously coincides with the space $D(A^\infty)$.

Theorem 4.4.3. *Let A be a symmetric operator in the Hilbert space H with domain $D(A)$. Then the following statements are valid:*

a) *If the operator A is self-adjoint, then the operator A^N is a self-adjoint operator in the Fréchet–Hilbert space H^N with a sequence of hilbertian seminorms* (4.4.3) *and with the domain $D(A)^N$.*

b) *The operator A^∞, defined by equality* (4.4.4) *on whole space $D(A^\infty)$, is a continuous self-adjoint operator in the space $D(A^\infty)$.*

c) *If the operator A has a pure point spectrum, then A^∞ has self-adjoint inverse operator $(A^\infty)^{-1}$ in the space $D(A^\infty)$. Further, if A is positive definite in H, then A^∞ is isomorphism of the space $D(A^\infty)$ on itself and the energetic space E_{A^∞} of the operator A^∞ coincides with $D(A^\infty)$.*

d) *If the positive operator A has a purely pointwise spectrum of non-decreasing positive eigenvalues $\{\lambda_k\}$, then A^∞ is positive defined selfadjoint isomorphism onto the space $D(A^\infty)$.*

Proof. a) It is not difficult to prove that Hilbert adjoint of the operator A^N in the Fréchet–Hilbert space H^N is $(A^*)^N$ with the domain of definition $D(A^*)^N$ and, therefore, it follows from the condition that the operator A^N is self-adjoint in H^N.

b) Obviously, the domain of definition of the operator A^∞ is whole space $D(A^\infty)$, i.e. notation of the space $D(A^\infty)$ is now also a notation for the domain of definition for the operator A^∞. Let $x, y \in D(A^\infty)$, then for each $n \in \mathbb{N}_0$ we have

$$\begin{aligned}\langle A^\infty x, y\rangle_n &= (Ax, y) + (A^2x, Ay) + \cdots + (A^n x, A^{n-1}y) \\ &= (x, Ay) + (Ax, A^2y) + \cdots + (A^{n-1}x, A^n y) = \langle x, A^\infty y\rangle_n.\end{aligned}$$

Hence, by virtue of statement a) of Theorem 4.2.2, we obtain that A^∞ is continuous self-adjoint operator in the space $D(A^\infty)$.

c) By Rellich's theorem ([160], p. 493), the self-adjoint operator in the Hilbert space H has a pure point spectrum if and only if the embedding of its domain $D(A)$ (with the norm $\|u\|_{D(A)} = \|Au\| + \|u\|$) in H is compact. It follows from this that the space $D(A^\infty)$ is a space of type (FS), i.e. projective limit of sequence of Hilbert spaces with compact embeddings.

In particular, in our case the space $D(A^\infty)$ is Montel, i.e. in it every closed and bounded set is compact. By virtue of Theorem 3 of [163], the operator A^∞ has a complete orthogonal system of the eigenelements $\{\operatorname{orb}(A, \varphi_k)\}$ in $D(A^\infty)$ (with the eigenvalues λ_k), which is also an unconditional basis in $D(A^\infty)$. (If the space $D(A^\infty)$ is nuclear, then this basis is absolute.) It follows from this that the image $R(A^\infty)$ of the operator A^∞ is everywhere dense in the space $D(A^\infty)$, since eigenelement for the operator A^∞ is contained in its image. Indeed, by virtue of the above isomorphism, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}A^\infty \operatorname{orb}(A, \varphi_k) &= A^\infty(\varphi_k, A\varphi_k, A^2\varphi_k, \dots) = (A\varphi_k, A^2\varphi_k, A^3\varphi_k, \dots) \\ &= (\lambda_k\varphi_k, \lambda_k A\varphi_k, \lambda_k A^2\varphi_k, \dots) = \lambda_k(\varphi_k, A\varphi_k, A^2\varphi_k, \dots) \\ &= \lambda_k A^\infty \operatorname{orb}(A, \varphi_k).\end{aligned}$$

Therefore, by virtue of statement c) of Theorem 4.2.2, we obtain that A^∞ has a self-adjoint inverse operator $(A^\infty)^{-1}$.

Let now A be positive definite in H, then for $x \in D(A^\infty)$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\langle A^\infty x, x\rangle_n &= \langle A^\infty \operatorname{orb}(A, x), \operatorname{orb}(A, x)\rangle_n \\ &= (Ax, x) + (A^2x, Ax) + \cdots + (A^n x, A^{n-1}x) \\ &\geq \gamma[(x, x) + (Ax, Ax) + \cdots + (A^{n-1}x, A^{n-1}x)] \\ &= \langle \gamma^2(\operatorname{orb}(A, x), \operatorname{orb}(A, x))\rangle_n = \gamma^2\langle x, x\rangle_n, \end{aligned} \tag{4.4.5}$$

i.e. A^∞ is also positive definite in $D(A^\infty)$. By virtue of the statement b) of Theorem 4.2.2, $(A^\infty)^{-1} \in L_0(E)$. Since, due to Theorems 4.2.3, $(A^\infty)^{-1}$ has a self-adjoint extension $\widetilde{(A^\infty)^{-1}}$ such that $D(\widetilde{(A^\infty)^{-1}}) = D(A^\infty)$, we get that

$\widetilde{(A^\infty)^{-1}} = (A^\infty)^{-1}$. Hence, A^∞ is an isomorphism of the space $D(A^\infty)$ onto itself.

Energetic space E_{A^∞} of the operator A^∞ is considered by the topology τ_{A^∞}, which is generated by a sequence of norms

$$\begin{aligned}[x]_n^2 &= \langle A^\infty x, x\rangle_n = \langle A^\infty \operatorname{orb}(A,x), \operatorname{orb}(A,x)\rangle_n \\ &= (Ax,x) + \cdots + (A^n x, A^{n-1}x). \end{aligned} \tag{4.4.6}$$

By virtue of (4.4.5) we have that $[x]_n^2 \geq \gamma^2\langle x,x\rangle_n$. Hence, the topology τ_{A^∞} is not weaker than the topology of the space $D(A^\infty)$ and therefore they coincide, i.e. $E_{A^\infty} = D(A^\infty)$. In particular, this means that the orbital equation

$$A^\infty \operatorname{orb}(A,u) = \operatorname{orb}(A,f) \tag{4.4.7}$$

has the unique stable solution in $D(A^\infty)$ for any right-hand side $\operatorname{orb}(A,u) \in D(A^\infty)$.

According to the statement b), A^∞ is defined on whole space $D(A^\infty)$ and is continuous. According to the statement c), A^∞ possesses a self-adjoint inverse operator $(A^\infty)^{-1}$ and is isomorphism onto the space $D(A^\infty)$. Let A possesses an orthogonal eigenfunctions φ_k corresponding to eigenvalues λ_k. In this case, A^∞ possesses the orthogonal eigenfunctions $\operatorname{orb}(A,\varphi_k)$ corresponding to the eigenvalues λ_k. Then for $x \in D(A^\infty)$, $\operatorname{orb}(A,x) = \sum\limits_{k=1}^{\infty} a_k \operatorname{orb}(A,\varphi_k)$ and by symmetry of A, the following relations hold:

$$\begin{aligned}\langle A^\infty x, x\rangle_n &= \Big\langle A^\infty \sum_{k=1}^{\infty} a_k \operatorname{orb}(A,\varphi_k), \sum_{k=1}^{\infty} a_k \operatorname{orb}(A,\varphi_k)\Big\rangle_n \\ &= \Big\langle \sum_{k=1}^{\infty} a_k A^\infty \operatorname{orb}(A,\varphi_k), \sum_{k=1}^{\infty} a_k \operatorname{orb}(A,\varphi_k)\Big\rangle_n \\ &= \Big\langle \sum_{k=1}^{\infty} \lambda_k a_k \langle \operatorname{orb}(A,\varphi_k), a_k \operatorname{orb}(A,\varphi_k)\rangle_n \\ &\geq \lambda_1 \|\operatorname{orb}(A,x)\|_n^2 = \lambda_1 \langle x,x\rangle_n, \end{aligned}$$

where $\lambda_1 = \min\{\lambda_k\}$ and $a_k = \frac{\langle \operatorname{orb}(A,x), \operatorname{orb}(A,\varphi_k)\rangle_n}{\langle \operatorname{orb}(A,\varphi_k), \operatorname{orb}(A,\varphi_k)\rangle_n} = \frac{(x,\varphi_k)}{(\varphi_k,\varphi_k)}$. □

Next, we prove that the extended Ritz method for equation (4.4.7) converges in the energetic Frecher space E_{A^∞} of the operator A^∞. The space E_{A^∞} in this case coincides with the Fréchet space $D(A^\infty)$ and its topology is stronger than the topology energetic Hilbert space H_A of the operator A.

The extended Ritz method is used for an approximate solution of the equation (4.4.7) in the Fréchet space $D(A^\infty)$, where A^∞ is again the self-adjoint operator with pure point spectrum. As basic functions the eigenfunctions of the operator A^∞ are chosen and it is proved that subspaces spanned by the first m eigenvector have orthogonal complement in the Fréchet space $D(A^\infty)$. This means what the approximate solutions do not depend on the number of norms generating the tonology of the space $D(A^\infty)$. It is proved the convergence of the sequences of approximation solutions in the space $D(A^\infty)$, whose topology is stronger than that of the original Hilbert space (Theorem 4.4.4). It is proved that the given in Theorem 4.4.5 algorithm is simultaneously central spline algorithm in the Fréchet space $D(A^\infty)$. Examples of self-adjoint and positive elliptical differential operators satisfying the conditions of Theorem 4.4.4 are considered. It is further indicated that the results obtained can be applied to the well-known differential operators of Lagrange, Lejandre and Tricomi.

The results of numerical experiments are presented, confirming justice theoretical research in the case of harmonic oscillator operator. For the last operator, the space $D(A^\infty)$ coincides with the Schwarz space $S(\mathbb{R})$ of rapidly decreasing functions [163]. Therefore, if as basis functions we take the eigenfunctions $\{\varphi_j\}$ of this operator, i.e. Hermite functions, then the sequence of approximate solutions converges to an exact solution in the Schwarz space $S(\mathbb{R})$, which topology is stronger that the topology of Sobolev space.

4.4.2 Application of the extended Ritz method for approximate solutions to an equation with a strongly degenerate elliptic operator

For an approximate solution of the equation (4.4.7) in the space $D(A^\infty)$ we apply extension of the Ritz method from Section 4.2 (approximate methods for solving of operator equations in Fréchet spaces were previously also considered in [134, 202]). We will only need a special case of this method and therefore we do not give its proof in the general case. As basis functions we choose the sequence of eigenfunctions $\{\varphi_j\}$ (resp. $\{\mathrm{orb}(A,\varphi_j)\}$) of operator A (resp. A^∞). System of equations for determining coefficients of approximate solutions for arbitrary $l \in \mathbb{N}$, takes the form

$$\sum_{k=1}^{m} [\varphi_k, \varphi_j]_{\,l} \alpha_k = (f, \varphi_j)_{\,l}, \quad j = 1, \dots, m,$$

i.e.

$$\sum_{k=1}^{m} \langle A^\infty \varphi_k, \varphi_j \rangle_l a_k = \langle f, \varphi_j \rangle_{\,l}, \quad j = 1, \dots, m,$$

where

$$
\begin{aligned}
[\varphi_k, \varphi_j]_{\,l} &= \langle A^{\infty}\varphi_k, \varphi_j\rangle_{\,l} = \langle A^{\infty}\operatorname{orb}(A, \varphi_k, \operatorname{orb}(A, \varphi_j)\rangle \\
&= (A\varphi_k, \varphi_j) + (A^{\,2}\varphi_k, A\varphi_j) + \cdots + (A^{l}\varphi_k, A^{\,l-1}\varphi_j) \\
&= (\lambda_k + \lambda_k^2\lambda_j + \cdots + \lambda_{\,k}^{\,l}\lambda_j^{\;l-1})(\varphi_k, \varphi_j).
\end{aligned} \tag{4.4.8}
$$

If the sequence $\{\varphi_j\}$ is orthogonal in H, from (4.4.8) we get that

$$
[\varphi_k, \varphi_j]_l = \begin{cases} 0, & \text{when } \; k \neq j, \\ [\varphi_j]_{\,l}^{\,2} = (\lambda_j + \lambda_j^3 + \cdots + \lambda_j^{\;2l-1}), & \text{when k=j;} \end{cases}
$$

In addition, we have

$$
\begin{aligned}
\langle f, \varphi_j\rangle_{\,l} &= \langle \operatorname{orb}(A, f), \operatorname{orb}(A, \varphi_j)\rangle_l \\
&= (f, \varphi_j) + (Af, A\varphi_j) + \cdots + (A^{\,l-1}f, A^{\,l-1}\varphi_j) \\
&= (f, \varphi_j) + \lambda_j^2(f, \varphi_j) + \cdots + \lambda_j^{2(l-1)}(f, \varphi_j) \\
&= (1 + \lambda_j^2 + \cdots + \lambda_j^{2(l-1)})(f, \varphi_j).
\end{aligned} \tag{4.4.9}
$$

It follows that

$$
\alpha_j = \frac{(1 + \lambda_j^2 + \cdots + \lambda_j^{\;2(l-1)})(f, \varphi_j)}{\lambda_j(1 + \lambda_j^2 + \cdots + \lambda_j^{\;2(l-1)})\|\varphi_j\|^2} = \frac{(f, \varphi_j)}{\lambda_j\|\varphi_j\|^2}\,.
$$

Therefore, the approximate solution takes the form

$$
\operatorname{orb}(A, u_m) = \sum_{j=1}^{m} \frac{(f, \varphi_j)}{\lambda_j\|\varphi_j\|^2}\operatorname{orb}(A, \varphi_j). \tag{4.4.10}
$$

If the sequence $\{\varphi_j\}$ is orthonormal in H, then

$$
\operatorname{orb}(A, u_m) = \sum_{j=1}^{m} \frac{(f, \varphi_j)}{\lambda_j}\operatorname{orb}(A, \varphi_j). \tag{4.4.11}
$$

It should be noted that the approximate solutions (4.4.11) do not depend on $l \in \mathbb{N}$. Therefore, by virtue of the classical Ritz theorem, this sequence converges to the solution of the equation (4.4.7) with respect to the energetic norm (4.4.6). Indeed, in this case the canonical maps $K_{A^{\infty},n} : E_{A^{\infty}} \to E_{A^{\infty}} / \operatorname{Ker}[\cdot]_n$ are identity mappings $I_n : D(A^{\infty}) \to (D(A^{\infty}), [\cdot]_n)$ defined by the equality

$$
I_n x = I_n(x, Ax, \ldots) = (x, Ax, \ldots, A^{\,n-1}x).
$$

The projection operators (4.2.1) have the form

$$A_n^\infty(I_n x) = I_n(A^\infty x) = I_n((Ax, A^2x, \dots,)) = (Ax, A^2x, \dots, A^n x)$$

and act in Hilbert spaces $(\widetilde{D(A^\infty)}, [\cdot]_n)$, where $(\widetilde{D(A^\infty)}, [\cdot]_n)$ is completion of the space $(D(A^\infty), [\cdot]_n)$. But the norms of $[\cdot]_n$ are isometric to the norms $[\cdot]_{A_n^\infty}$ for each $n \in \mathbb{N}$, where

$$[I_n x]_{A_n^\infty} = \langle A_n^\infty I_n x, I_n x\rangle_n^{1/2} = ((Ax, x) + \cdots + (A^n x, A^{n-1}x))^{1/2}.$$

Therefore, the sequence $\{\mathrm{orb}(A, u_m)\}$ converges to $(A^\infty)^{-1} f = \mathrm{orb}((A^\infty)^{-1}, f)$ in the energetic space E_{A^∞} and therefore in the space $D(A^\infty)$. This proves that the following theorem takes place.

Theorem 4.4.4. *Let $A : D(A) \subset H \to H$ be a self-adjoint and positive definite operator in the Hilbert space H with an orthogonal sequence of eigenfunctions $\{\varphi_j\}$. Then the sequence of approximate solutions $\{u_m\}$, constructed by the Ritz method* (4.4.10)*, converges to the solution of the equation* (4.4.7) *in the space $D(A^\infty)$.*

Next, we will show that the algorithm, constructed in Theorem 4.4.4 for approximate solutions to the equation (4.4.7) in the Fréchet space $D(A^\infty)$, is simultaneously central and spline algorithm. For the metrization of the space $D(A^\infty)$ we will use the metric (4.3.7), constructed for nondecreasing sequences of norms $\{\,[\,\cdot\,]_n\,\}$.

Indeed, let $I(f) = [L_1(f), \cdot, L_m(f)]$ be non-adaptive information of cardinality m, where $L_i(f) = (f, \varphi_i)$, $i = 1, \dots, m$. The subspace $\mathrm{Ker}I$ is a finite-codimensional subspace in the energetic space E_{A^∞}. Therefore, $\mathrm{Ker}I$ has the topological complement $\mathrm{Ker}I^\perp = \mathrm{span}\{\varphi_1, \dots, \varphi_m\}$ in E_{A^∞}. Let $e_i = (0, \dots, 1, \dots)$, where 1 is in the i-th place. Then $((\varphi_i, \varphi_i)/||\varphi_i||^2)\varphi_i \in I^{-1}(e_i)$ and the best approximation of the element φ_i in $\mathrm{Ker}I^\perp$ is the same as φ_i. It follows that the spline interpolatory e_i is φ_i and the interpolatory spline $y = I(f)$ has the form $\sigma = \sum_{i=1}^n (f, \varphi_i)||\varphi_i||^{-2}\varphi_i$. The solution operator for equation (4.3.3) is $S = (A^\infty)^{-1}$, which realized an isomorphism of the space E_{A^∞} onto itself. We have

$$\begin{aligned} S\sigma &= \sum_{i=1}^n (f, \varphi_i)||\varphi_i||^{-2} S\varphi_i = \sum_{i=1}^n (f, \varphi_i)||\varphi_i||^{-2}(A^\infty)^{-1}\varphi_i \\ &= \sum_{i=1}^n \lambda_i^{-1}(f, \varphi_i)||\varphi_i||^{-2}\varphi_i = u_m, \end{aligned}$$

where u_m is an approximate solution of equation (4.3.3), constructed using the extended Ritz method. $S\sigma = u_m$ is also an element of the best approximation $Sf = (A^\infty)^{-1}f$ in the subspace $\mathrm{Ker}I^\perp$ with respect to the energetic norms $[\cdot]_n$ of the energetic space E_{A^∞} of the operator A^∞. The subspace $\mathrm{Ker}I$ has an orthogonal complement $\mathrm{Ker}I^\perp$ in E_{A^∞}. According to Theorem 3.6.2, the spline algorithm φ^s, defined by the equality $\varphi^s(I(f)) = u_m$, is central.

It follows from this that it is fair

Theorem 4.4.5. *Let A be a self-adjoint and positive operator with discrete spectrum in the Hilbert space H with an orthogonal sequence of eigenfunctions $\{\varphi_i\}$. Let λ_i be the positive eigenvalue corresponding to the eigenfunction $\{\varphi_i\}$ and let u_m be the approximate solution of the equation $A^\infty u = f$ defined by equality* (4.4.7). *Then the algorithm $\varphi^s(I(f)) = u_m$ is a linear spline and central algorithm for the operator $S = (A^\infty)^{-1}$ and information $I(f) = [(f, \varphi_1), \cdot, (f, \varphi_m)]$.*

Moreover, the sequence of approximate solutions $\{u_m\}$ converges to a solution of equation (4.4.7) in the space E_{A^∞} and in the space $D(A^\infty)$.

The space $D(A^\infty)$, with a nondecreasing sequence of hilbertian norms $\{||\cdot||_n\}$ is isomorphic to the projective limit of the sequence of Hilbert spaces $(D(A^n), ||\cdot||_n)$ with respect to mappings

$$\pi_{mn} : \ (D(A^m), ||\cdot||_m) \to (D(A^n), ||\cdot||_n) \ \ (n \le m)$$

and

$$K_n : \ D(A^\infty) \to (D(A^n), ||\cdot||_n), \ \ n \in \mathbb{N},$$

where

$$\pi_{mn} : \ (x, Ax, A^2x, \dots, A^{m-1}x) \to (x, Ax, A^2x, \dots A^{n-1}x)$$

and

$$K_n : \ (x, Ax, A^2x, \dots) \to (x, Ax, \dots, A^{n-1}x).$$

The Fréchet spaces $D(A^\infty)$ are important for studying the distribution of moduli of eigenvalues of self-adjoint elliptic boundary value problems [160].

Let H be a complex separable Hilbert space and A a be a self-adjoint operator with pure point spectrum acting on H. For such operators, the eigenvalues (counting with multiplicities) can be numbered in increasing order of modules:

$$0 \le |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_j| \le |\lambda_{j+1}| \le \cdots \to \infty \quad \text{as} \ \ j \to \infty.$$

The function

$$N(\lambda) = \sum_{|\lambda_j| \le \lambda} 1$$

gives the number of eigenvalues not exceeding the absolute value of the number λ.

Below s_j are the approximation numbers of the operator A, defined as follows:

$$s_n(A, D(A), H) = s_n(A) = \inf \{\|A - K\|;\ \dim R(K) \leq n \text{ and } K \in L(E)\},$$
$$n = 0, 1, \dots,$$

where $R(K)$ is the image of the operator K.

Theorem 4.4.6 ([160], Section 5.6.1)**.** *Let A be a self-adjoint operator with pure point spectrum and let $\varkappa > 0$. The following statements are equivalent:*

$$\begin{aligned} &\text{a)} \quad N(\lambda) + 1 \sim \lambda^{\varkappa} + 1, \quad \lambda \geq 0; \\ &\text{b)} \quad 1 + |\lambda_j| \sim j^{1/\varkappa}, \quad j = 1, 2, \dots; \\ &\text{c)} \quad s_j(I; D(A), H) \sim j^{-1/\varkappa}, \quad j = 1, 2, \dots . \end{aligned}$$

(*The sign $\sim$ means that the right-hand side can be estimated from above and below through the left-hand side multiplied by some constants independent of λ or j, respectively.*)

Theorem 4.4.7 ([160], Section 8.2.2)**.** *Let H be a separable Hilbert space and A be a self-adjoint operator on H.*

a) *The space $D(A^\infty)$ is Montel if and only if the operator A has a pure point spectrum.*

b) *The space $D(A^\infty)$ is nuclear if and only if the operator A has a pure point spectrum and there exist numbers $c > 0$ and $\tau > 0$ such that*

$$N(\lambda) \leq c\lambda^\tau + 1.$$

c) *The space $D(A^\infty)$ is isomorphic to the space s of rapidly decreasing sequences if and only if the operator A has a pure point spectrum and there are numbers $c_1 > 0$, $c_2 > 0$, $\tau_1 > 0$ and $\tau_2 > 0$ such that*

$$c_1\lambda^{\tau_1} + 1 \leq N(\lambda) + 1 \leq c_2\lambda^{\tau_2} + 1.$$

Theorem 4.4.8 ([160], Section 8.3.1)**.** *Let $\Omega \subset R_n$ be an arbitrary domain and $\overset{\circ}{A}$, defined by*

$$\overset{\circ}{A}u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u, \quad D(\overset{\circ}{A}) = C_0^\infty(\Omega),$$

be symmetric operator in $L_2(\Omega)$ with coefficients $a_\alpha(x)$ of class C^∞. Suppose further that there exists a self-adjoint extension A of the operator $\overset{\circ}{A}$ and that $D(A^\infty)$ is a nuclear Fréchet space. Then $D(A^\infty)$ is isomorphic to the space s of rapidly decreasing sequences.

The isomorphism between the spaces $D(A^\infty)$ and s is carried out by the mapping

$$f \to \{(f, \varphi_j)\}_{j=1}^{\infty},$$

where $\{\varphi_j\}$ forms a basis in $D(A^\infty)$. For special spaces $\overline{C}^\infty((a,b))$, $\overline{C}_0^\infty((a,b))$, $S(\mathbb{R})$ and others, there are many explicit bases that generate an isomorphic mapping on s. Thus, Mityagin [108] proved that the Chebyshev polynomials form a basis in $\overline{C}^\infty((-1,1))$. Guillemot-Teissier [67] showed that Legendre polynomials also form a basis in $\overline{C}^\infty((-1,1))$, since they are eigenfunctions of the classical Legendre differential operator. The Hermite functions form a basis in $S(\mathbb{R})$. This statement also follows from the considerations below, since the Hermite functions serve as eigenfunctions of the Hermite differential operator, which is a harmonic oscillator operator. However, a number of important nuclear spaces are not covered by the above method. Examples include $C_0^\infty(\mathbb{R})$, $C^\infty(\mathbb{R})$, as well as nuclear spaces of harmonic and analytic functions. (Definitions of these spaces are available from A. Pietsch [125].) Mityagin [108] proved that $C^\infty(\mathbb{R})$ is isomorphic to $(s)^N$. This structural result and Theorem 4.4.8 show that $C^\infty(\mathbb{R})$ cannot be represented as $D(A^\infty)$, where A is a self-adjoint differential operator on $L_2(\mathbb{R})$. Let us finally note a corollary from one of Rosenblum's results ([160], Section 8.3.3), in which the distributions of eigenvalues of the self-adjoint polyharmonic differential operator A with Dirichlet boundary conditions in unbounded domains are studied. Rosenblum proved that there exists a domain Ω for which the operator A has a pure point spectrum in $L_2(\Omega)$ such that (contrary to the usual behavior of eigenvalue distributions of differential operators)

$$N(\lambda) \sim e^{c\lambda}, \quad c > 0.$$

This implies that the space $D(A^\infty)$ is a Montel space, but not a nuclear space.

4.4.3 Examples of second order differential operators

Example 1. Let H be a Hilbert space with the inner product $(\,\cdot\,)$, S be a self-adjoint and positive definite operator from H to H with a sequence of eigenvectors $\{\varphi_i\}$, F_1-Fréchet space $D(S^\infty)$ with the sequence of hilbertian norms

$$[x]_n^2 = (S^\infty x, x)_n = (Sx, x) + (S^2x, Sx) + \cdots + (S^{n+1}x, S^n x), \quad n \in \mathbb{N}_0,$$

where $V_n = \{x \in D(S^\infty); [x]_n \leq 1\}$, information

$$If = [\,(Sf, \varphi_1), \ldots, (Sf, \varphi_m)] = (y_1, \ldots, y_m),$$

and $\operatorname{Ker} I = \{x \in D(A^\infty); [x, \varphi_1] = 0, \ldots, [x, \varphi_m] = 0\}$. His orthogonal subspace $\operatorname{Ker} I^\perp = \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$. $\operatorname{Ker} I$ has orthogonal complement in the

Fréchet space $D(S^{\infty})$, since the best approximation of the element $u \in D(S^{\infty})$ in $\operatorname{Ker} I^{\perp}$ does not depend on n, and the spline σ is the best approximation of the element u in $\operatorname{Ker} I^{\perp}$. Hence, the conditions of Theorem 4.4.5 are met and, therefore, there is a spline algorithm for the solutions to the equation $S^{\infty}u = f$, which is central. In fact, we have information $(y_1, \dots, y_m) = [\,(Su, \varphi_1), \dots, (Su, \varphi_m)]$ and find the element of best approximation σ of the element u in subspace $\operatorname{Ker} I$, which has an orthogonal complement $\operatorname{Ker} I^{\perp}$, spanned by $\varphi_1, \dots, \varphi_m$. Therefore, the spline σ is the best approximation for u in $\operatorname{Ker} I^{\perp}$ with respect to all energetic norms $[\,\cdot\,]_n$ and coincides with an approximate solution constructed by using the Ritz method.

In particular, according to the results of Section 3.5, the conditions Theorem 4.4.5 satisfy:

a) Information I in the strict Fréchet–Hilbert spaces F_1, for which the subspace $\operatorname{Ker} I$ is again the strict Fréchet–Hilbert space.

b) Information I in the countable-Hilbert spaces with the nondecreasing sequence of norms $\{\|\cdot\|_n\}$, which is generated by continuous functionals of the space $(F_1, \|\cdot\|_1)'$.

Let us now give the examples of symmetric and positive differential operators satisfying the conditions of Theorem 4.4.4. These examples are taken mainly from the works [**?**, 160].

Example 2. For an arbitrary region $\Omega \subset \mathbb{R}^l$, through $C^{\infty}(\Omega)$, as usual, denotes the space of all infinitely differentiable functions, defined in Ω. Next, let $\rho(x) \in C^{\infty}(\Omega)$ be a positive function such that

1. For any multi-index γ there is $C_\gamma > 0$ such that $|D^{\gamma}\rho(x)| \le C_\gamma \rho^{\,1+|\gamma|}(x)$, for all $x \in \Omega$.

2. For any $K > 0$ there are the numbers $\varepsilon_k > 0$ and $r_k > 0$ such that $\rho(x) > K$ if $d(x) \le \varepsilon_k$, or $|x| \ge r_k$, when $x \in \Omega$ ($d(x)$ is distance from x to the boundary $\partial(\Omega)$).

Let $S_{\rho(x)}(\Omega)$ denote a metrizable locally convex space

$$S_{\rho(x)}(\Omega) = \{f \in C^{\infty}(\Omega);\ \ \|f\|_{n,\alpha} = \sup\ \rho^n(x)|D^{\alpha}f(x)| < \infty,$$
$$\text{for every } \ n \in \mathbb{N}_0 \ \text{ and all multi-indexes } \ \alpha\}. \tag{4.4.12}$$

It should be noted that for each bounded domain Ω there exists a function $\rho(x)$ for which $\rho^{-1}(x)$ essentially coincides with $d(x)$.

It is known ([160], Section 6.2.3) that the space $S_{\rho(x)}(\Omega)$ is the Fréchet space which contains $C_0^\infty(\Omega)$ as dense subspace. In what follows, we will consider the spaces $S_{\rho(x)}(\Omega)$, which are contained in the space $L^p(\Omega)$ $(1 \leq p < \infty)$. This is equivalent to the condition: there exists $a > 0$, for which $\rho^{-a}(x) \in L^1(\Omega)$. Due to Theorem ([160], Section 6.2.3), we have that the space $S_{\rho(x)}(\Omega)$ is a nuclear Fréchet space isomorphic to the space s of rapidly decreasing sequences. Below it will be shown that the known Schwarz space $S(\mathbb{R})$ represents a special case of such spaces.

Let $\Omega \subset \mathbb{R}^l$ be an arbitrary region and $\rho(x)$ be the above weight function. Further, let $r \in \mathbb{N}$, and μ and ν be real numbers, and $\nu > \mu + 2r$. Set

$$\chi_q = \frac{1}{2\,r}(\nu(2\,r - q) + \mu q), \quad q = 0, 1, \ldots, 2r.$$

The class $\mathfrak{R}^r_{\mu,\nu}(\Omega, \rho(x))$ consists of all differential operators of the form

$$Au = \sum_{q=0}^{m} \sum_{|j|=2q} \rho^{\chi_{2q}}(x) b_\alpha(x) D^\alpha u + \sum_{|\beta|<2r} \alpha_\beta(x) D^\beta u.$$

Here $b_\alpha(x) \in C^\infty(\Omega)$ $(|\alpha| = 2q$, where $q = 0, 1, \ldots, r)$ are real functions, all derivatives of which (including themselves) are bounded in Ω. In addition, it is assumed that there exists a positive number C such that for all $\xi \in R^n$ and $x \in \Omega$,

$$(-1)^r \sum_{|\alpha|=2\,r} b_\alpha(x)\xi^\alpha \geq C|\xi|^{2r}, \quad b_{(0,\ldots,0)}(x) \geq C,$$
$$(-1)^q \sum_{|\alpha|=2\,q} b_\alpha(x)\xi^\alpha \geq 0, \quad q = 1, \ldots, r-1$$

(ellipticity condition). Finally, $\alpha_\beta(x) \in C^\infty(\Omega)$ $(0 \leq |\beta| < 2r)$ and $D^\gamma \alpha_\beta(x) = o(\rho^{\chi_{|\beta|}+|\gamma|})$ for any multi-index γ.

The class $\mathfrak{R}^r_{\mu,\nu}(\Omega, \rho(x))$ is quite wide class of degenerate elliptic differential operators ([160], Section 6.2.1). Let's give an example of an operator from this class. Operator A given by the relations

$$Au = -\Delta u + \rho^\nu(x) u, \quad \nu > 2, \quad D(A) = C_0^\infty(\Omega)\,, \tag{4.4.13}$$

is essentially self-adjoint in $L^2(\Omega)$, i.e. its closure $\overline{A}$ is a self-adjoint operator in $L^2(\Omega)$, $D(\overline{A}) = W_2^2(\Omega\,,1\,,\rho^{2\nu})$ ([160], Section 6.4.1) and $\overline{A}$ has pure point range.

Moreover, A is positive definite and, by Theorem 4.4.2, the spectrum of $\overline{A}$ is discrete. Subsequence of the eigenfunctions $\{\varphi_j\}$ of the operator $\overline{A}$ belong to the space $S_{\rho(x)}(\Omega)$. It is also proved that $D(\overline{A}^j) = W_2^j(\Omega\,,1\,,\rho^{2\nu j})$ ([160], Section

6.4.3) and the space $S_{\rho(x)}(\Omega)$ is isomorphic to the space $D(\overline{A}^{\infty})$, where the topology of the space $D(\overline{A}^{\infty})$ is given by a sequence of hilbertian norms (4.4.1), and the topology of the space $S_{\rho(x)}(\Omega)$ is given by the sequence $\{\,\|\cdot\|_{n,\alpha}\}$. Therefore, if we consider the equation

$$-\Delta u + \rho^{\nu}(x)u = f \tag{4.4.14}$$

in the Fréchet space $S_{\rho(x)}(\Omega)$ with a sequence of norms (4.4.12), then, by virtue of statement b) of Theorem 4.4.3 (see also [160], Section 6.4.3), it has unique solution for each $f \in S_{\rho(x)}(\Omega)$. If the sequence of eigenfunctions $\{\varphi_j\}$ is orthogonal in the space $L^2(\Omega)$, then the sequence of approximate solutions $\{u_m\}$, built by the Ritz method and given by the equality (4.4.10), converges in the space $S_{\rho(x)}(\Omega)$ to the solution of equation (4.4.14).

Let us now give a specification of this result in the one-dimensional case for harmonic oscillator operator

$$Au = -u''(t) + t^2 u$$

without additional boundary conditions. It is self-adjoint ([111], p. 387) and positive definite operator in the Hilbert space $L^2(\mathbb{R})$. From Molchanov's theorem ([16], p. 393) it follows that this operator has a purely discrete spectrum. Therefore, according to Theorem 4.4.2, this spectrum is discrete. According to [126], the space $D(A^{\infty})$ for the operator A is the Schwarz space $S(\mathbb{R})$. Eigenfunctions of the harmonic oscillator operator are Hermite functions (wave functions of harmonic oscillator) ([129], p. 115), since

$$\left(-\frac{d^2}{dt^2} + t^2\right)\varphi_j = (2j+1)\varphi_j\,,$$

where

$$\varphi_0 = \pi^{-1/4} e^{\frac{t^2}{2}}$$

and

$$\varphi_j(t) = (2^j j!)^{-1/2}(-1)^j \pi^{-1/4} e^{\frac{t^2}{2}} \left(\frac{d}{dt}\right)^{(j)} e^{-t^2} \quad (j \geq 1)\,. \tag{4.4.15}$$

This means that $\lambda_j = 2j+1$, $j = 1, 2, \ldots$. Subsequence $\{\varphi_j\}$ is an orthonormal basis in the space $L^2(\mathbb{R})$ and, by virtue of the nuclearity of the space $S(\mathbb{R})$, an absolute basis in it. We consider the space $S(\mathbb{R})$ with a sequence of hilbertian norms

$$\|h\|_n = \left(\|h\|^2 + \left\| \left(-\frac{d^2}{dt^2} + t^2 \right) h \right\|^2 + \cdots + \left\| \left(-\frac{d^2}{dt^2} + t^2 \right)^{n-1} h \right\|^2 \right)^{1/2}, \tag{4.4.16}$$

where $\|\cdot\|$ is the norm of the space $L^2(\mathbb{R})$.

Let the operator A^∞ be the restriction of the operator A to the space $S(\mathbb{R}) \subset D(A)$ taking into account the topology of the last space. By virtue of (4.4.11), the approximate solution of the equation $A^\infty \operatorname{orb}(A, u) = \operatorname{orb}(A, f)$ has the form

$$\operatorname{orb}(A, u_m) = \sum_{j=1}^{m-1} \frac{(f, \varphi_j)}{2j+1} \operatorname{orb}(A, \varphi_j), \tag{4.4.17}$$

where $\{\varphi_j\}$ are defined by the equality (4.4.15). The sequence of the approximate solutions $\{\operatorname{orb}(A, u_m)\}$ converges to the solution of the equation $A^\infty \operatorname{orb}(A, u) = \operatorname{orb}(A, f)$ in the topology of space $S(\mathbb{R})$.

Jointly with S. A. Razmadze it is composed a program realizing the convergence of the sequence of approximate solutions for various functions from the space $S(\mathbb{R})$. The obtained numerical results confirmed the above theoretical conclusions. Numerical analogues of the approximate solutions (4.4.17) converge to the solution of the equation $A^\infty \operatorname{orb}(A, u) = \operatorname{orb}(A, f)$ for enough large norm numbers from (4.4.16) for various functions from the space $S(\mathbb{R})$.

The results obtained can be applied to essentially self-conjugate and positive definite operators of Legendre $A_{m,k}$ $(2k \leq m)$ ([160], Section 7.4.1) and Tricomi $B_{n,k}$ ([160], Section 7.6.3). There are also given the representations of the spaces $D(\overline{A}^\infty_{m,k})$ ([160], Section 7.4.4) and $D(\overline{B}^\infty_{m,k})$ ([160], Section 7.6.3).

Example 3. Consider the differential operator B from ([**?**], p. 106), which has the form

$$Bu = -\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{\nu^2}{x}u\right], \quad \nu^2 = const > 0, \quad 0 < x < 1.$$

We will consider it as an operator in the weighted space $H = L_2(x; 0, 1)$ of functions that are quadratically summable with weight x on the interval $(0, 1)$. The domain of definition of $D(B)$ consists of functions of the space $L^2(x; 0, 1)$ for which $u(x)$ and $u'(x)$ are absolutely continuous on any segment of the form $[\varepsilon, 1]$ $(0 < \varepsilon < 1)$; the product $\sqrt{x}\,u'(x)$ is continuous on the segment $[0, 1]$ and vanishes at $x = 0$; $Bu \in H$ and $u(1) = 0$. In ([**?**], p. 107), it is proved that $D(B)$ is dense in H, B is symmetric, positive definite in H, and has a discrete spectrum. The generalized eigenvalues of the operator B have the form

$$\lambda_k = j^2_{\nu,k}, \quad k \in \mathbb{N}, \tag{4.4.18}$$

$$\lambda_k = j_{\nu,k}^2, \quad k \in \mathbb{N}, \tag{4.4.19}$$

where $j_{\nu,k}$ is the k-th positive root of the Bessel function $\mathcal{J}_\nu(x)$, corresponding to the normalized eigenfunctions

$$\varphi_k(x) = \frac{\sqrt{2}}{\mathcal{J}_{\nu+1}(j_{\nu,k})}\,\mathcal{J}_\nu(j_{\nu,k}x), \quad k = 1, 2, \dots . \tag{4.4.20}$$

It is known that

$$\mathcal{J}_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{m!\Gamma(\nu+m+1)}$$

and the series on the right-hand side converges in the whole plane of the complex variable z (with the possible exception of $z = 0$) and allows member-by-member differentiation. Hence, for $\nu \geq 0$, it turns out that $\mathcal{J}_\nu(z) = O(z^\nu)$ and $\mathcal{J}_\nu'(z) = O(z^{\nu-1})$. Therefore, if $\nu > \frac{1}{2}$, for the function $\varphi_k(x)$ defined according to (4.4.20), we have $\sqrt{x}\,\varphi_k'(x) \to 0$ for $x \to 0$, i.e. the generalized eigenfunctions $\varphi_k(x)$ of the operator B are in fact its ordinary eigenfunctions.

Approximate solutions $\operatorname{orb}(B, u_m)$ of the equation $B^\infty \operatorname{orb}(B, u) = \operatorname{orb}(B, f)$ in the Fréchet space $D(B^\infty)$, constructed using the generalized Ritz method, have the form

$$\operatorname{orb}(B, u_m) = \sum_{k=1}^{m} \lambda_k^{-1} \int_0^1 x f(x)dx \ \operatorname{orb}(B, \varphi_k(x)),$$

where λ_k and φ_k are defined according to (4.4.18) and (4.4.20). The sequence u_m converges in the space $D(B^\infty)$ to a solution of the equation $B^\infty \operatorname{orb}(B, u) = \operatorname{orb}(B, f)$ if $f \in D(B^\infty)$.

Example 4. Let's consider the Sturm–Liouville operator ([**?**], p. 101)

$$Au = -\frac{d}{dx}\left[p(x)\,\frac{du}{dx}\right] + q(x)u \tag{4.4.21}$$

in the space $L_2[a, b]$. We define the operator A on the set $D(A)$ of functions continuous on the segment $[a, b]$, having the absolutely continuous first derivative and a square-integrable second derivative, under boundary conditions

$$u(a) = u(b) = 0. \tag{4.4.22}$$

On the functions $p(x)$ and $q(x)$ we impose the conditions $p, p', q \in C[a, b]$, $p(x) \geq p_0 = c > 0$, $q(x) \geq 0$. It is known that A is a positive definite operator. The norms in H_A and in $\overset{\circ}{W}{}_2^{(1)}(a,b)$ are equivalent. Since embedding of the space H_A into $L_2(a,b)$ is completely continuous, the spectrum of the operator A is discrete.

Finding the spectrum of the operator A considered here is equivalent to the problem, which is called the Sturm–Liouville problem: find values of the parameter λ for which nontrivial solutions of the equation

$$\frac{d}{dx}\left(p(x)\,\frac{du}{dx}\right) - q(x)u + \lambda u = 0 \quad (Au = \lambda u),$$

exist satisfying the boundary conditions (4.4.22).

In some special cases, the Sturm-Liouville problem is solved effectively:

Example 4.1. $p(x) \equiv 1$, $q(x) \equiv 0$.

The question comes down to finding those values of λ for which the differential equation

$$\frac{d^2u}{dx^2} + \lambda u = 0 \quad \left(A = -\frac{d^2}{dx^2}\right)$$

has a nontrivial solution satisfying the conditions

$$u(a) = u(b) = 0.$$

It is proved ([**?**], p. 105) that the eigenvalues λ_k and eigenfunctions $\varphi_k(x)$ are given by the formulas

$$\lambda_k = \frac{k^2\pi^2}{(b-a)^2}\,, \quad k \in \mathbb{N}\,,$$

$$\varphi_k(x) = \sqrt{\frac{2}{b-a}}\,\sin\frac{\pi k(x-a)}{b-a}\,, \quad k \in \mathbb{N}\,.$$

Approximate solutions of the equation $A^\infty\,\mathrm{orb}(A,u) = \mathrm{orb}(A,f)$ in the Fréchet spaces $D(A^\infty)$ have the form (4.4.11).

Example 4.2. Consider the operator

$$A = -\frac{d^2}{dx^2} + I \quad \left(Au = -\frac{d^2u}{dx^2} + u\right)$$

under the boundary conditions

$$u'(a) = u'(b) = 0.$$

This operator is positive definite, its eigenvalues are

$$\lambda_k = 1 + \frac{k^2\pi^2}{(b-a)^2}\,, \quad k \in \mathbb{N}\,,$$

and the corresponding eigenfunctions are equal to

$$u_0(x) = 0, \quad \varphi_k(x) = \sqrt{\frac{2}{b-a}} \cos \frac{k\pi(x-a)}{b-a}, \quad k \in \mathbb{N}. \tag{4.4.23}$$

Approximate solutions of the equation $A^\infty \operatorname{orb}(A, u) = \operatorname{orb}(A, f)$,

$$-\frac{d^2u}{dx^2} + u = f,$$

in the space $D(A^\infty)$, constructed using the generalized Ritz method, have the form (4.4.11).

Example 5. Consider the operator

$$Au = -\frac{d^2u(x)}{dx^2} + q(x)u(x)$$

in the space $H = L_2(0,1)$. Let $q(x)$ be a real function continuous on $[0,1]$ and $D(A)$ be the set of all functions $u(x)$ with the following properties: $u(x)$ and $u'(x)$ are absolutely continuous for $x \in [0,1]$, $u(0) = u(1) = 0$ and $u''(x) \in L_2(0,1)$. It is easy to verify that A is a symmetric operator and $D(A)$ is dense in H. Suppose that $q(x)$ is such that the equation $Au = 0$ has no solution equal to zero at $x = 0$ and $x = 1$, other than the trivial $u \equiv 0$. In this case, it is proved that $R(A) = H$, which implies that A is a self-adjoint operator. If $q(x) \geq 0$, then A will be a positive definite operator with discrete spectrum ([112], pp. 81, 102). For $q(x) = x^2$, this operator was studied above.

Example 6. Beltrami operator δ.

Let S be the unit sphere of l-dimensional Euclidean space $\mathbb{R}^l$, and $\vartheta_1, \vartheta_2, \dots, \vartheta_{l-1}$ be the spherical coordinates of the point $\theta \in S$. On the sphere S we define the class of functions $C^{(2)}(S)$ as follows. Let $f(\theta)$ be a function defined on S. Let us construct a spherical layer $\Sigma = \{x : \rho_1 \leq |x| \leq \rho_2,\, x \in \mathbb{R}^l\}$, where ρ_1 and ρ_2 are arbitrary, but fixed positive numbers. We can assume that $\rho_1 < 1 < \rho_2$, so $S \subset \Sigma$. We extend the function $f(\theta)$ to the layer Σ using the formula $f^*(x) = f\left(\frac{x}{|x|}\right)$. It is clear that the extended function $f^*(x)$ is constant on any ray passing through the origin, and therefore does not depend on $\rho = |x|$. The class $C^{(2)}(S)$ is defined as follows: $f \in C^{(2)}(S)$ if $f^* \in C^{(2)}(\Sigma)$.

The operator δ is defined on the set $C^{(2)}(S)$ as follows:

$$\delta = -\sum_{j=1}^{l-1} \frac{1}{q_j \sin^{l-j-1} \vartheta_j} \frac{\partial}{\partial \vartheta_j} \left(\sin^{l-j-1} \vartheta_j \frac{\partial}{\partial \vartheta_j} \right),$$

where $q_1 = 1$, $q_j = (\sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{j-1}^2)^2$, $j \geq 2$.

In the space $H = L_2(S)$ this operator is symmetric and the values $\lambda_n = n(n+l-2)$, $n = 1, 2, \dots$, are its eigenvalues of the multiplicity

$$k_{n,l} = (2n+l-2)\,\frac{(l+n-3)!}{(l-2)!n!}\,.$$

The eigenfunctions corresponding to the eigenvalue λ_n are the spherical functions $Y_{n,l}^{(k)}(\theta)$, $1 \leq k \leq k_{n,l}$. The operator δ has no other eigenvalues and eigenfunctions ([**?**], p. 239). $\{Y_{n,l}^{(k)}\}$ is an orthonormal complete system in $L_2(S)$. Since all eigenvalues of λ_n are positive, we conclude that δ^∞ is a positive definite operator, and the spectrum of λ_n is discrete.

We number the spherical functions $Y_{n,l}^{(k)}$ in the following way. It is assumed that $l \geq 2$. If $1 \leq k \leq k_{1,l} = l$, then we take $\lambda_k = l(l-1)$; $\phi_k(\theta) = Y_{1,l}^{(k)}(\theta)$ and if $k_{1,l} + \cdots + k_{j,l} < k \leq k_{1,l} + \cdots + k_{j+1,l}$, then $\lambda_k = (j+1)(j+l-1)$; $\phi_k(\theta) = Y_{j+1,l}^{(k-(k_{1,l}+\cdots+k_{j,l}))}(\theta)$. If we substitute these λ_k and ϕ_k in (4.4.10), then we will obtain a sequence $\{u_m\}$ for the approximate solution of the equation $\delta^\infty\,\mathrm{orb}(\delta, u) = \mathrm{orb}(\delta, f)$. For such a sequence, Theorem 4.4.5 is valid in the space $D(\delta^\infty)$. Then the algorithm $\phi^s(I(f)) = u_m$ is a linear spline and central algorithm for the operator $(\delta^\infty)^{-1}$ and information $I(f) = [(f, \phi_1), \cdots, (f, \phi_m)]$.

CHAPTER 5

Stability and spline algorithms for computerized tomography problems

The problem of determining the solution to an equation of the form $Au = f$ with the operator A, acting from the space E into a similar space F, is called correctly posed on a pair of spaces (E, F) if the following conditions are fulfilled: 1) the image $R(A)$ of the operator A coincides with F (solvability condition); 2) the solution is determined uniquely; 3) the inverse of the operator A is continuous on F (stability condition). This concept belongs to J. Hadamard. He also owns a classic example of an ill-posed problem – the Cauchy problem for the Laplace equation. As it later turned out, the need to solve this particular problem arises in a wide variety of areas of mathematics and natural science. Ill-posed problems are: solutions of integral equations of the 1-st kind; differentiation of functions known approximately; numerical summation of Fourier series; analytical continuation of functions; inverse problems of gravimetry; a number of biophysical problems; supersonic flow around bodies, etc. Ill-posed problems include a wide class of so-called inverse problems arising in physics, technology and other branches of knowledge, in particular, the problem of processing the results of physical experiments. Recently, ill-posed problems of X-ray computed tomography, which have important applications in diagnostic medicine and in problems of testing the strength of materials, have been intensively studied.

The development of the theory and methods for solving ill-posed (unstable) problems is associated with the names of prominent mathematicians A. N. Tikhonov, G. I. Marchuk, M. M. Lavrentiev, V. K. Ivanov, F. Netterer, A. Louis and others.

A. N. Tikhonov owns the following generalization of the classical (according to Hadamard) concept of correctness, which is based on the fundamental idea of restriction the domain of definition of the original operator [156]. Namely, the problem $Au = f$ is called Tikhonov-correct (conditionally correct) if 1) it is known a priori that a solution exists for some class of data from F and belongs to the given set M; 2) the solution is unique in the class M; 3) infinitesimal variations of the right-hand side f that do not remove the solution from the class M correspond to infinitesimal variations of the solution.

This chapter considers the ill-posed equation $Ku = f$ in the Hilbert space H for a compact self-adjoint operator K with positive eigenvalues. It is assumed that the conditions of existence and uniqueness are fulfilled, but the stability condition is not satisfied, i.e. the inverse operator K^{-1} is not continuous. In [156], for some ill-posed problems, a metric compact space E is considered, which the operator K maps onto itself isomorphically. Therefore, such equations in the space E have a unique stable solution. Similarly, we carry over the above incorrect equation to the Fréchet space E, in which the operator K is an isomorphism of the space E onto itself. More precisely, the Fréchet space E, as a set, is a part of H and the restriction K to the Fréchet space E is a self-adjoint operator on E that maps the space E isomorphically onto itself. For an approximate solution of the resulting equation in the Fréchet space E, we use the generalized Ritz method from Sections 4.3–4.4. A condition is given under which this method is a spline algorithm. The obtained results are used to construct a spline algorithm for an approximate solution of the computerized tomography problem.

5.1 Ritz method for the approximate solution of equations with compact operators

Let K be a linear, self-adjoint, positive, injective and compact operator in the Hilbert space H with dense image. Let $\{\varphi_k\}$ be some orthogonal sequence of eigenelements of the operator K and λ_k be the sequence of eigenvalues corresponding φ_k, $k \in \mathbb{N}$. Then K has the form $Ku = \sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi_k)^{-1} (u, \varphi_k)\varphi_k$, where $\lambda_k \to 0,\ \lambda_k > 0$. In Section 1.5, it is proved that $\{\varphi_k\}$ is a complete system in H, the inverse of K^{-1} to the operator K is self-adjoint, positive definite and has the form

$$K^{-1}x = \sum_{k=1}^{\infty} \lambda_k^{-1} (x, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k \,.$$

The sequence λ_k^{-1} is unbounded and tends to infinity. Therefore, K^{-1} is a self-adjoint operator with a discrete spectrum and dense domain of definition.

The space $D(K^{-\infty})$ and operator K_∞. Consider the Fréchet space $D(K^{-\infty}) = \bigcap_{n=0}^{\infty} D(K^{-n})$ with the hilbertian norms

$$\|f\|_n^2 = \|f\|^2 + \|K^{-1}f\|^2 + \cdots + \|K^{-n}f\|^2, \quad n \in \mathbb{N}_0,$$

which are generated by the inner products

$$\langle x, y\rangle_n = (x, y) + (K^{-1}x, K^{-1}y) + \cdots + (K^{-n}x, K^{-n}y), \quad n \in \mathbb{N}_0,$$
$$x, y \in D(K^{-\infty}), \ \ n \in \mathbb{N}_0.$$

As is known, an arbitrary Fréchet space is isomorphic to the subspace of the product of a sequence of Banach spaces. In our case, the Fréchet space $D(K^{-\infty})$ is isomorphic to the subspace M of the Fréchet space H^N and this isomorphism is realized by the mapping

$$\begin{aligned} x \in D(K^{-\infty}) &\to \operatorname{Orb}(K^{-1}, x) \\ &= (x, K^{-1}x, K^{-2}x, \ldots, K^{-n}x, \ldots) \in M \subset H^{\mathbb{N}_0}. \end{aligned} \tag{5.1.1}$$

This means that the Fréchet space $D(K^{-\infty})$ is isomorphic to the space of all orbits of the operator K^{-1} at the point x. The topology of the Fréchet space $H^{\mathbb{N}_0}$ is given by the following seminorms:

$$\|f\|_n^2 = \|f_0\|^2 + \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2, \quad f = \{f_k\} \in H^{\mathbb{N}_0},$$

which are generated by the semi-scalar products

$$\langle x, y\rangle_n = (x_0, y_0) + (x_1, y_1) + (x_2, y_2) + \cdots + (x_n, y_n),$$
$$x = \{x_k\}, \ \ y = \{y_k\} \in H^{\mathbb{N}_0}, \ \ n \in \mathbb{N}_0.$$

Let us define the operator $K^{-\infty} : D(K^{-\infty}) \to D(K^{-\infty})$ according to the equality

$$K^{-\infty}x = \{K^{-1}x, K^{-2}x, \ldots, K^{-n}x, \ldots\}.$$

This operator is continuous because it is defined on the entire Fréchet space $D(K^{-\infty})$. It is symmetric and positive definite on the Fréchet space $D(K^{-\infty})$, since for an arbitrary $n \in \mathbb{N}_0$ and $x, y \in D(K^{-\infty})$, the equality

$$\langle K^{-\infty}x, y\rangle_n = \langle x, K^{-\infty}y\rangle_n$$

and the inequalities

$$[x]_n^2 = \langle K^{-\infty}x, x\rangle_n \geq C_n\langle x, x\rangle_n \tag{5.1.2}$$

are valid, where

$$[x]_n^2 =: \langle K^{-\infty}x, x\rangle_n = (K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n-1}x, K^{-n}x),$$

i.e. $[x]_n$ are the norms of the energetic space $E_{K^{-\infty}}$ of the operator $K^{-\infty}$. Let us prove the inequality (5.1.2). Indeed, we have

$$\begin{aligned}[x]_1^2 &= \langle K^{-\infty}x, x\rangle_0 = (K^{-1}x, x)\\ &= \left(\sum_{k=1}^{\infty}\lambda_k^{-1}(x,\varphi_k)\varphi_k, \sum_{k=1}^{\infty}(x,\varphi_k)\varphi_k\right) = \sum_{k=1}^{\infty}\lambda_k^{-1}(x,\varphi_k)^2 \geq C_0\langle x, x\rangle_0,\end{aligned}$$

where $C_0 = \min\{\lambda_k^{-1};\ k\in\mathbb{N}_0\}$ and

$$\begin{aligned}\langle K^{-\infty}x, x\rangle_1 &= (K^{-1}x, x) + (K^{-2}x, K^{-1}x)\\ &\geq C_0(x,x) + \Big(\sum_{k=1}^{\infty}\lambda_k^{-2}(x,\varphi_k)\varphi_k, \sum_{k=1}^{\infty}\lambda_k^{-1}(x,\varphi_k)\varphi_k)\\ &= C_0(x,x) + \sum_{k=1}^{\infty}\lambda_k^{-3}(x,\varphi_k)^2 \geq C_0(x,x) + C_0\sum_{k=1}^{\infty}\lambda_k^{-2}(x,\varphi_k)^2\\ &\geq C_0(x,x)_2.\end{aligned}$$

The inequality (5.1.2) is proved similarly for arbitrary $n\in\mathbb{N}_0$.

So, in the Fréchet space $D(K^{-\infty})$, there are two sequences of norms

$$\|f\|_n^2 = \|f\|^2 + \|K^{-1}f\|^2 + \cdots + \|K^{-n+1}f\|^2, \quad n\in\mathbb{N}_0,$$

and

$$\begin{aligned}[f]_n =: \langle K^{-\infty}f, f\rangle_n^{1/2} = ((K^{-1}f, f) + (K^{-2}f, K^{-1}f) + \cdots\\ +(K^{-n-1}f, K^{-n}f))^{1/2}, \quad n\in\mathbb{N}_0.\end{aligned}$$

From the inequality (5.1.2) it follows that the second sequence of norms generates a stronger topology in the Fréchet space $D(K^{-\infty})$ and, therefore, coincide.

It is clear that for the basis sequence $\{\varphi_k\}$ of the Hilbert space H, the sequence $\{\mathrm{orb}(K^{-1},\varphi_k)\}$ is also eigenelement of the operator $K^{-\infty}$. Therefore, the operator $K^{-\infty}$ has a dense image in the Fréchet space $D(K^{-\infty})$ and the operator $(K^{-\infty})^{-1}$ exists and is continuous by statement b) of Theorem 4.2.2. The operator $(K^{-\infty})^{-1}$ is self-adjoint, since the operator $K^{-\infty}$ is self-adjoint (statement c) of Theorem 4.2.2). Therefore, the operator $K^{-\infty}$ is an isomorphism of the Fréchet space $D(K^{-\infty})$ onto itself and the equation $(K^{-\infty})^{-1}u = f$ in the Fréchet space $D(K^{-\infty})$ has a unique stable solution. Let us denote the operator $(K^{-\infty})^{-1}$ by K_∞ (the notation K_∞, and not K^∞, was chosen to distinguish this operator from the operator A^∞ considered in Section 4.4). The operator $K_\infty = (K^{-\infty})^{-1}$ coincides with the restriction of the operator K^N from the Fréchet space H^N to the

Fréchet space $D(K^{-\infty})$. Note that

$$K_\infty(u) = (K^{-\infty})^{-1}(u) = (Ku, KK^{-1}u, \dots, KK^{-n+1}u, \dots)$$
$$= (Ku, u, K^{-1}u, \dots, K^{-n+2}u, \dots)$$

and therefore

$$K^{-\infty}K_\infty(\operatorname{orb}(K^{-1}, u)) = K_\infty(K^{-\infty}\operatorname{orb}(K^{-1}, u))$$
$$= K_\infty(K^{-1}u, K^{-2}u, \dots, K^{-n}u, \dots)$$
$$= (u, K^{-1}u, \dots, K^{-n}u, \dots) = \operatorname{orb}(K^{-1}, u).$$

The equation $Ku = f$ in the Hilbert space H is ill-posed and for $f \in D(K^{-\infty})$ we transfer it from H into the Fréchet space $D(K^{-\infty})$. Namely, f must be replaced by $\operatorname{orb}(K^{-1}, f) = (f, K^{-1}f, \dots)$ and Ku by $K_\infty u = \operatorname{orb}(K^{-1}, Ku) = (Ku, u, K^{-1}u, \dots)$. For simplicity, we write the transferred equation in the form

$$K_\infty u = f \tag{5.1.3}$$

or

$$K_\infty \operatorname{orb}(K^{-1}, u) = \operatorname{orb}(K^{-1}, f),$$

hoping that this agreement will not lead to misunderstandings. Due to the fact that K_∞ is an isomorphism of $D(K^{-\infty})$ onto itself, the equation (5.1.3) has a unique and stable solution in the Fréchet space $D(K^{-\infty})$, i.e. it is correct.

We appeared in a similar situation as A. Tikhonov in his work [156]. In this work, in particular, for some ill-posed problems, a metric compact space E was considered, which the operator K mapped isomorphically onto itself. Therefore, such equations in the space E have a unique stable solution. Similarly, the ill-posed problem

$$Ku = f,$$

with the self-adjoint operator K considered above, we transferred to the Fréchet space $E = D(K^{-\infty})$. The restriction K_∞ of the operator K to this space is a self-adjoint operator. Moreover, K_∞ is an isomorphism of the Fréchet space $D(K^{-\infty})$ onto itself and therefore the equation (5.1.3) in the Fréchet space $D(K^{-\infty})$ has stable solution.

Consider $D(K^{-\infty})$ equipped with a sequence of hilbertian norms

$$\{x\}_n = \langle K_\infty x, x\rangle_n^{1/2} = ((Kx, x) + (KK^{-1}x, K^{-1}x)$$
$$+ \cdots + (K^{-n+1}x, K^{-n}x))^{1/2}, \quad n \in \mathbb{N}_0, \tag{5.1.4}$$

which is generated by the inner products

$$\{x,y\}_n = \langle K_\infty x, y\rangle_n = (Kx,y) + (KK^{-1}x, K^{-1}y) \\ + \cdots + (K^{-n+1}x, K^{-n}y), \quad n \in \mathbb{N}_0.$$

According to (5.1.4), we get that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\{x\}_n^2 &= (Kx,x) + (KK^{-1}x, K^{-1}x) + \cdots + (K^{-n+1}x, K^{-n}x) \\ &\geq (Kx,x) + C_0\|x\|^2 + C_0\|K^{-1}x\|^2 + \cdots + C_0\|K^{-n+1}x\|^2 \\ &= (Kx,x) + C_0\|x\|_{n-1}^2 \geq C_0\|x\|_{n-1}^2.\end{aligned}$$

This also means that the sequences of norms $\{\{\cdot\}_n\}$ and $\{\|\cdot\|_n\}$ generate comparable topologies on $D(K^{-\infty})$. The Fréchet space $D(K^{-\infty})$, equipped with a sequence of hilbertian norms $\{\{\cdot\}_n\}$, we call the energetic space of the operator K_∞ and denote it by E_{K_∞}. Therefore, the Fréchet spaces E_{K_∞} and $D(K^{-\infty})$ are isomorphic. To approximately solve the equation, we use the extended Ritz method in the space E_{K_∞}. Coefficients of the approximate solution $u_m = \sum_{k=1}^m a_k\varphi_k$ are determined from the following system of equations:

$$\sum_{k=1}^m a_i\{\varphi_k, \varphi_i\}_n = (f, \varphi_k)_n, \ \ i = 1,2,\ldots,m, \ \ n \in \mathbb{N}_0,$$

i.e.

$$\sum_{k=1}^m a_i(K_\infty\varphi_k, \varphi_i)_n = (f, \varphi_k)_n, \ \ i = 1,2,\ldots,m, \ \ n \in \mathbb{N}_0.$$

In general, if $\{\varphi_k\}$ is an arbitrary linearly independent orthogonal sequence, the coefficients a_k depend on n. Now we will prove that, in our case, they do not depend on n. Indeed, we have

$$\begin{aligned}\langle K_\infty\varphi_k, \varphi_i\rangle_n &= (K\varphi_k, \varphi_i) + (KK^{-1}\varphi_k, K^{-1}\varphi_i) + \cdots + (K^{-n+1}\varphi_k, K^{-n}\varphi_i) \\ &= \lambda_k(\varphi_k, \varphi_i)(1 + \lambda_k^{-1}\lambda_i^{-1} + \cdots + \lambda_k^{-n}\lambda_i^{-n}),\end{aligned}$$

i.e.

$$\langle K_\infty\varphi_k, \varphi_i\rangle_n = \begin{cases} 0 & \text{if } \ k \neq i, \\ \lambda_k(\varphi_k, \varphi_k)(1 + \lambda_k^{-2} + \cdots + \lambda_k^{-2n}) & \text{if } \ k = i. \end{cases}$$

Besides,

$$\begin{aligned}\langle f, \varphi_k\rangle_n &= (f, \varphi_k) + (K^{-1}f, K^{-1}\varphi_k) + \cdots + (K^{-n}f, K^{-n}\varphi_k) \\ &= (f, \varphi_k) + (f, K^{-2}\varphi_k) + \cdots + (f, K^{-2n}\varphi_k) \\ &= (1 + \lambda_k^{-2} + \cdots + \lambda_k^{-2n})(f, \varphi_k).\end{aligned}$$

Therefore, it turns out that

$$a_k = \lambda_k^{-1}(f, \varphi_k)(\varphi_k, \varphi_k)^{-1}.$$

Hence the approximate solution of equation (5.1.3) obtained by using extended Ritz method takes the form

$$\operatorname{orb}(K^{-1}, u_m) = \sum_{k=1}^{m} (f, \varphi_k)((\varphi_k, \varphi_k)\lambda_k)^{-1} \operatorname{orb}(K^{-1}, \varphi_k). \tag{5.1.5}$$

Let $y = I(f) = [L_1(f),\ L_2(f), \dots, L_m(f)]$ be a non-adaptive information of cardinality m on $D(K^{-\infty})$, where $L_i(f) = (f, \varphi_i)$, $i, \dots, m$. $\operatorname{Ker} I$ is a subspace of finite codimension in $D(K^{-\infty})$ and $(\operatorname{Ker} I)^{\perp} = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Spline σ_m interpolatory y has the shape

$$\sigma_m = \sum_{k=1}^{m} (f, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k. \tag{5.1.6}$$

The solution operator for the equation $K_\infty u = f$ is $S = (K_\infty)^{-1} = K^{-\infty}$ and it realizes an isomorphism of the space $D(K^{-\infty})$ onto itself. From (5.1.6) it follows that

$$\begin{aligned} S\sigma_m &= \sum_{k=1}^{m} (f, \varphi_k)(\varphi_k, \varphi_k)^{-1} S\varphi_k = \sum_{k=1}^{m} (f, \varphi_k)(\varphi_k, \varphi_k)^{-1} (K_\infty)^{-1} \varphi_k \\ &= \sum_{k=1}^{m} (f, \varphi_k)(\varphi_k, \varphi_k)^{-1} K^{-\infty} \varphi_k = \sum_{k=1}^{m} \lambda_k^{-1} (f, \varphi_k)(\varphi_k, \varphi_k)^{-1} \varphi_k = u_m, \end{aligned}$$

since

$$\begin{aligned} K^{-\infty}(\varphi_k) &= (K^{-1}\varphi_k, K^{-2}\varphi_k, \dots, K^{-n}\varphi_k, \dots) = \operatorname{orb}(K^{-1}, K^{-1}\varphi_k) \\ &= \lambda_k^{-1}(\varphi_k, K^{-1}\varphi_k, \dots, K^{-1}\varphi_k, \dots) = \lambda_k^{-1} \operatorname{orb}(K^{-1}, \varphi_k) = \lambda_k^{-1}\varphi_k. \end{aligned}$$

$S\sigma_m = u_m$ is the best approximation element for $Sf = (K_\infty)^{-1} f$ in the subspace $(\operatorname{Ker} I)^{\perp}$ with respect to the energetic norms $\{\cdot\}_n$ of the energetic space E_{K_∞} of the operator K_∞ for all $n \in \mathbb{N}_0$.

Indeed, the unique best approximation element for $Sf = (K_\infty)^{-1} f$ in the subspace $(\operatorname{Ker} I)^{\perp}$ with respect to the hilbertian norm $\{\cdot\}_n$ in the pre-Hilbert space

$(E_{K_\infty}, \{\cdot\}_n)$ has the form

$$\begin{aligned}\sum_{k=1}^{m}&\{(K_\infty)^{-1}f,\varphi_k\}_n\{\varphi_k,\varphi_k\}_n^{-1}\varphi_k\\ &=\sum_{k=1}^{m}\langle K_\infty(K_\infty)^{-1}f,\varphi_k\rangle_n\{\varphi_k,\varphi_k\}_n^{-1}\varphi_k\\ &=\sum_{k=1}^{m}(f,\varphi_k)((\varphi_k,\varphi_k)\lambda_k)^{-1}\varphi_k=u_m.\end{aligned}$$

Therefore, this element of the best approximation does not depend on n.

The best approximation of this kind in locally convex spaces has been considered by many mathematicians (see review in [132]). It follows that the subspace G_m spanned by the functions $\varphi_1,\dots,\varphi_m$ has an orthogonal complement $\operatorname{Ker} I$ in E_{K_∞}. It should be noted that the orthogonality in Fréchet spaces significantly differs from the orthogonality in Hilbert spaces (see Section 2.4).

From the completeness of the system φ_k it follows that the sequence of constructed best approximations elements $\{u_m\}$ tends to $(K_\infty)^{-1}f$ in the energetic space E_{K_∞} (see Theorem 4.4.5). Indeed, we will use the same scheme that was used to prove Theorem 4.3.4. In this case, we also define the canonical mappings, which are the identity mappings J_n defined by the following equalities:

$$J_n(x)=J_n\operatorname{orb}(K^{-1},x)=\operatorname{orb}_n(K^{-1},x),\quad n\in\mathbb{N}_0.$$

The projective operators of the operator K_∞ have the form

$$\begin{aligned}K_{\infty,n}(J_n(x))&=J_n(K_\infty(x))=J_n(Kx,x,K^{-1}x,K^{-2}x,\dots,K^{-n+1}x,\dots)\\ &=(Kx,x,K^{-1}x,K^{-2}x,\dots,K^{-n+1}x).\end{aligned}$$

They map the Hilbert space $(\widetilde{D(K^{-\infty})},\{\cdot\}_n)$ into itself, where $(\widetilde{D(K^{-\infty})},\{\cdot\}_n)$ is the completion of the pre-Hilbert space $(D(K^{-\infty}),\{\cdot\}_n)$. In this space $(D(K^{-\infty}),\{\cdot\}_n)$, we define the energetic space $(D(K^{-\infty}),\{\cdot\}_n)_{K_{\infty,n}}$ of the operator $K_{\infty,n}$ and the energetic norm according to the formula

$$\begin{aligned}[J_n(x)]_{K_{\infty,n}}&=\langle K_{\infty,n}(J_n(x)),J_n(x)\rangle_n=((Kx,x)+(KK^{-1}x,K^{-1}x)\\ &\qquad+(KK^{-2}x,K^{-2}x)+\cdots+)+(K^{-n+1}x,K^{-n}x))^{1/2}\\ &=((Kx,x)+(x,K^{-1}x)+(K^{-1}x,K^{-2}x)+\cdots\\ &\qquad+(K^{-n+1}x,K^{-n}x))^{1/2},\ \ n\in\mathbb{N}.\end{aligned}$$

It is clear that $\{\cdot\}_n$ are isometric to the norms of $[\cdot]_{K_{\infty,n}}$ and the spaces $(D(K^{-\infty}), \{\cdot\}_n)_{K_{\infty,n}}$ are isometric to the spaces $(D(K^{-\infty}), \{\cdot\}_n)$ for all $n \in \mathbb{N}_0$.

According to the classical Ritz Theorem (only for a positive operator), we find that for all $n \in \mathbb{N}_0$, the sequence of approximate solutions $\{u_m\}$ converges to the element $(K_\infty)^{-1} f$ with respect to the norm $[\cdot]_{K_{\infty,n}}$ in the space $(D(K^{-\infty}), \{\cdot\}_n)_{K_{\infty,n}}$ for all n, where it represents a sequence of partial sums. Therefore, due to isometricity, it also converges in the Fréchet space E_{K_∞}.

On the other hand, the following inequalities are valid:

$$
\begin{aligned}
[f]_n &= ((K^{-1}f, f) + (K^{-2}f, K^{-1}f) + \cdots + (K^{-n-1}f, K^{-n}f))^{1/2} \\
&\le \{f\}_{n+1} = ((Kf, f) + (f, K^{-1}f) + (KK^{-2}f, K^{-2}f) \\
&\qquad + \cdots + (K^{-n}f, K^{-n+1}f))^{1/2}, \quad n \in \mathbb{N}_0.
\end{aligned}
$$

This means that the sequence of norms $\{\{\cdot\}_n\}$ on the space $D(K^{-\infty})$ defines a stronger topology than $\{[\cdot]_n\}$.

From the above it follows that the following statement is true.

Theorem 5.1.1. *Let $K : H \to H$ be a compact, self-adjoint, positive and injective operator with dense image and orthogonal sequence of eigenelements φ_j. Let λ_j be the eigenvalues corresponding to the eigenelements φ_j and u_m are defined according to* (5.1.5). *Then the algorithm $\varphi^s(I(f)) = u_m$ is linear spline for the solution operator $S = K_\infty^{-1}$ and information $I(f) = [(f, \varphi_1), (f, \varphi_2), \cdots (f, \varphi_m)]$. In addition, the sequence of approximate solutions $\{u_m\}$ converges to the solution of equation* (5.1.3) *in the energetic space E_{K_∞} of the operator K_∞ with respect to the norms* (5.1.4)*, as well as in the space $D(K^{-\infty})$.*

We now give several examples of self-adjoint and positive definite operators in the Hilbert space from Section 1.5 for which the operator $K^{-\infty}$ satisfies the conditions of Theorem 5.1.1.

5.1.1 Application for some differential and integral operators

1. Inverse of QHO

For the QHO from Section 4.3, the inverse selfadjoint and positive operator $K = A^{-1}$ in the space $L^2(-\infty, \infty)$ has the form

$$
K(u) = \sum_{k=1}^{\infty} (2k+1)^{-1} (u, \varphi_k) \varphi_k.
$$

For this operator K, consider the equation $K_\infty u = f$ in the space $D(K^{-\infty}) = D(A^\infty) = S(\mathbb{R})$. In this case, the energetic space E_{K_∞} for K_∞ is $S(\mathbb{R})$. For this equation, the spline algorithm will be $\sum_{k=1}^{m}(2k+1)(f,\varphi_k)\varphi_k = u_m$. According to statement b) of Theorem 3.6.2, this algorithm is linear in the space E_{K_∞} with the sequence of energetic norms.

2. Integral equations of the first kind

The discussed Examples 2.1–2.3 below are compiled according to Examples 2.1, 2.6 and 2.11 of the second chapter of the third part of the book [77].

2.1. Consider the following integral equation of the first kind:

$$Ku(t) = \int_a^b K(s,t)u(s)\,ds = f(t), \tag{5.1.7}$$

where

$$K(s,t) = \begin{cases} (s-a)(t-b)(a-b)^{-1}, & a \le s \le t \le b, \\ (t-a_)(s-b)(a-b)^{-1}, & a \le t \le s \le b. \end{cases}$$

It is well known [77] that $K(s,t)$ is the Green function for the symmetric and positive operator $A = -d^2/dt^2$ in the Hilbert space $L^2[a,b]$ with boundary conditions $f(a) = f(b) = 0$. $D(A)$ is a set of functions having absolutely continuous first-order derivatives and square-summable second-order derivatives on $[a,b]$. $D(A^\infty)$ consists of functions having square-summable derivatives of infinite order on $[a,b]$. This space coincides with the countably normed Sobolev space of infinite order $W^\infty[a,b]$ (see Section 2.6). The operator K_∞, that is the restriction of the integral operator K on the space $D(A^\infty) = D(K^{-\infty})$, is a topological isomorphism and the equation (5.1.7) has a unique and stable solution. The eigenvalues and corresponding orthonormal eigenfunctions of the operator A are $\lambda_k = k^2\pi^2/(b-a)^2$ and $\varphi_k(t) = \sqrt{\frac{2}{b-a}}\sin\frac{\pi k(t-a)}{b-a}$, $k \in \mathbb{N}$. An approximate solution of the equation (5.1.7) has the following form:

$$\begin{aligned} &\operatorname{orb}(K^{-1}, u_m(t)) \\ &\quad = \sum_{k=1}^{m} \frac{2k^2\pi^2}{(b-a)^2} \int_a^b f(s)\sin\frac{\pi k(s-a)}{b-1}\,ds\ \operatorname{orb}\left(K^{-1}, \sin\frac{\pi k(t-a)}{b-a}\right). \end{aligned}$$

For this sequence, the above reasoning is valid and the sequence $\{u_m\}$ converges in the space E_{K_∞} to a solution of the equation (5.1.7). According to Theorem 5.1.1, this algorithm is linear and spline.

2.2. Consider the integral equation of the first kind (5.1.7), where

$$K(s,t) = \begin{cases} (e^s + e^{2a-s})(e^t + e^{2b-t})2^{-1}(e^{2b} - e^{2a})^{-1}, & a \le s \le t \le b, \\ (e^t + e^{2a-t})(e^s + e^{2b-s})2^{-1}(e^{2b} - e^{2a})^{-1}, & a \le t \le s \le b. \end{cases}$$

It is known that $K(s,t)$ is the Green function for the symmetric and positive operator $Af = -d^2f/dt^2 + f$ in the Hilbert space $L^2[a,b]$ with the boundary condition $f'(a) = f'(b) = 0$. $D(A)$ is a set of functions having absolutely continuous first-order derivatives and square-summable second-order derivatives on $[a,b]$. $D(A^\infty)$ consists of functions that have square-summable derivatives of infinite order on $[a,b]$. This space coincides with the countably normed Sobolev space of infinite order $W^\infty[a,b]$ (see Section 2.4). The operator K_∞, that is, the restriction of the integral operator K on the space $D(A^\infty) = D(K^{-\infty})$, is a topological isomorphism and the equation (5.1.7) has a unique and stable solution. The eigenvalues and corresponding orthonormal eigenfunctions of the operator A are $\lambda_k = 1 + k^2\pi^2/(b-a)^2$ and $\varphi_k(t) = \sqrt{\frac{2}{b-a}} \cos \frac{\pi k(t-a)}{b-a}$, $k \in \mathbb{N}$.

An approximate solution to the equation (5.1.7) has the following form:

$$\operatorname{orb}(K^{-1}, u_m(t)) = \sum_{k=1}^{m} \left(1 + \frac{2k^2\pi^2}{(b-a)^2}\right) \frac{2}{b-a} \int_a^b f(s) \cos \frac{\pi k(s-a)}{b-1}\, ds$$
$$\times \operatorname{orb}\left(K^{-1}, \cos \frac{\pi k(t-a)}{b-a}\right).$$

For this sequence, the above reasoning is valid and the sequence $\{\operatorname{orb}(K^{-1}, u_m(t))\}$ converges in the space E_{K_∞} to a solution of the equation (5.1.7). According to Theorem 5.1.1, this algorithm is linear and spline.

2.3. Consider the integral equation of the first kind (5.1.7), where $a = -\infty$, $b = \infty$ and

$$K(s,t) = \begin{cases} -\pi^{1/2} I(-\infty, s) I(t, \infty) \exp \frac{s^2+t^2}{2}, & s \le t, \\ -\pi^{1/2} I(s, \infty) I(-\infty, t) \exp \frac{s^2+t^2}{2}, & s \ge t, \end{cases}$$

where $I(u,v) = \int_u^v e^{-t^2}\, dt$. It is known that $K(s,t)$ is the Green function for the symmetric and positive definite degenerate hyperelliptic operator $Af(t) = -d^2f/dt^2 + (t^2+1)f$ in the Hilbert space $L^2(\mathbb{R})$ with the boundary condition $f(-\infty) = f(\infty) = 0$. $D(A)$ is a set of functions having absolutely continuous first-order derivatives and square-summable second-order derivatives on $(-\infty, \infty)$. $D(A^\infty)$ consists of functions that have summable derivatives of infinite order on

$(-\infty,\infty)$. This space contains a countable Hilbert–Sobolev space of infinite order $W^\infty(\mathbb{R})$ [21]. The operator K_∞, that is, the restriction of the integral operator K on the space $D(A^\infty) = D(K^{-\infty})$, is a topological isomorphism and the equation (5.1.7) has a unique and stable solution. The eigenvalues and corresponding orthonormal eigenfunctions of the operator A are $\lambda_k = 2k$ and $\varphi_k(t) = (-1)^{k-1}(k-1)^{-1/4}((k-1)!)^{-1/2}\pi^{-1/4}2^{1-k}e^{t^2/2}\,\frac{d^{k-1}e^{-t^2}}{dt^{k-1}}$, $k \in \mathbb{N}$. Using these functions φ_k we construct an approximate solution to the equation (5.1.7)

$$\operatorname{orb}(K^{-1}, u_m(t)) = 2\sum_{k=1}^{m} k \int\limits_{-\infty}^{\infty} f(s)\varphi_k(s)\,ds\ \operatorname{orb}(K^{-1}, \varphi_k(t)).$$

For this sequence, the above reasoning is valid and this sequence $\{u_m\}$ converges in the space E_{K_∞} to a solution of the equation (5.1.7). According to statement b) of Theorem 5.1.1, this algorithm is linear and spline.

5.2 Approximate solution of equations containing operators admitting SVD

Let H and M be Hilbert spaces and let $\{\varphi_k\}$, $\{\psi_k\}$ be orthogonal systems in H and M, respectively. Further, let A be an operator acting from H to M having a singular decomposition:

$$Au = \sum_{k=1}^{m} \sigma_k(u,\varphi_k)\psi_k\,, \quad u \in H, \quad \sigma_k > 0. \tag{5.2.1}$$

These operators in the space $D((A^*A)^{-n})$ were considered in Section 1.6. This generalized solution satisfies the equation

$$A^*Au = A^*f \tag{5.2.2}$$

and have the form

$$u^+ = \sum_{k=1}^{\infty} (\sigma_k(\psi_k,\psi_k)(\varphi_k,\varphi_k))^{-1}(f,\psi_k)\varphi_k\,. \tag{5.2.3}$$

Consider the Fréchet space $D((A^*A)^{-1})^\infty := D((A^*A)^{-\infty})$ with the sequence of norms

$$\|u\|_n = (\|u\|_n^2 + \|(A^*A)^{-1}u\|_n^2 + \cdots + \|(A^*A)^{-n}u\|_n^2)^{1/2}, \quad n \in \mathbb{N}_0,$$

which are generated by the inner products

$$\langle u,v\rangle_n = (u,v) + ((A^*A)^{-1}u, A^*A)^{-1}v) + \cdots + ((A^*A)^{-n}u, (A^*A)^{-n}v),$$
$$u,v \in D((A^*A)^{-\infty}), \quad n \in \mathbb{N}_0.$$

The restriction of the operator $(A^*A)^{-1}$ on the Fréchet space $D((A^*A)^{-\infty})$ coincides with the restriction of the operator $(A^*A)^{-N}$ from the space H^N to the Fréchet space $D((A^*A)^{-\infty})$.

By virtue of (5.1.1), we have the isomorphism

$$\begin{aligned} D((A^*A)^{-\infty}) \ni u &= (u, (A^*A)^{-1}u, (A^*A)^{-2}u, \dots) \\ &= \operatorname{orb}\big((A^*A)^{-1}, u\big) \in M \subset H^N. \end{aligned}$$

The definition of the orbital operator $(A^*A)^{-\infty}$ takes the form

$$\begin{aligned} (A^*A)^{-\infty} \operatorname{orb}((A^*A)^{-1}, u) &= ((A^*A)^{-1}u, (A^*A)^{-2}u, \dots) \\ &= \operatorname{orb}\big((A^*A)^{-1}, (A^*A)^{-1}u\big). \end{aligned}$$

Its inverse operator $(A^*A)_\infty = ((A^*A)^{-\infty})^{-1}$ takes the form

$$\begin{aligned} (A^*A)_\infty \operatorname{orb}((A^*A), u) &= ((A^*A)u, u, (A^*A)^{-1}u, (A^*A)^{-2}u, \dots) \\ &= \operatorname{orb}\big((A^*A)^{-1}u, (A^*A)u\big). \end{aligned}$$

Moreover, since the operator $(A^*A)^{-\infty}$ is self-adjoint and positive definite in the countable Hilbert space $D((A^*A)^{-\infty})$, it acts isomorphically from the Fréchet space $D((A^*A)^{\infty})$ to itself. Therefore, the operator $(A^*A)_\infty$ is also an isomorphism from the Fréchet space $D((A^*A)^{-\infty})$ onto itself, i.e. the equation

$$(A^*A)_\infty u = f, \quad f \in D((A^*A)^{-\infty}),$$

has a unique stable solution in Fréchet space $D((A^*A)^{-\infty})$. From here we also obtain that the equation

$$(A^*A)_\infty u = A^*g, \quad A^*g \in D((A^*A)^{-\infty}), \tag{5.2.4}$$

has the unique stable solution.

Let us consider the space $D((A^*A)^{-\infty})$ with a sequence of energetic norms of the operator $(A^*A)_\infty$, which have the form

$$\begin{aligned} \{u\}_n = \langle (A^*A)_\infty u, u\rangle_n^{1/2} &= ((A^*Au, u) + (u, (A^*A)^{-1}u) \\ &+ \cdots + ((A^*A)^{-n+1}u, (A^*A)^{-n}u), \quad n \in \mathbb{N}_0. \end{aligned}$$

If the sequences $\{\varphi_k\}$ and $\{\psi_k\}$ are orthogonal and

$$\lim_{k\to\infty} \sigma_k(\varphi_k, \varphi_k)(\psi_k \psi_k) = 0\,, \tag{5.2.5}$$

then A is compact. A^*A is also compact and the formula (5.1.5) can be applied. The approximate solution takes the form

$$\begin{aligned} u_m &= \sum_{k=1}^{m} (\sigma_k^2(\varphi_k,\varphi_k)(\psi_k,\psi_k)^2)^{-1}(A^*f,\varphi_k)\varphi_k \\ &= \sum_{k=1}^{m} (\sigma_k^2(\psi_k,\psi_k)(\varphi_k,\varphi_k)^2)^{-1}(f,A\varphi_k)\varphi_k \\ &= \sum_{k=1}^{m} (\sigma_k(\psi_k,\psi_k)(\varphi_k,\varphi_k))^{-1}(f,\psi_k)\varphi_k\,. \end{aligned}$$

This means that the approximate solution u_m of the equation (5.2.2) coincides with m-th partial sum of a generalized solution in the Moore–Penrose sense represented by equality (5.2.3). Let us take into account this remark and replace in Theorem 5.1.1 operator K by A^*A, where A has singular decomposition (5.2.1). In this case, it is valid

Theorem 5.2.1. *Let H and M be Hilbert spaces and let A be an operator with singular decomposition* (5.2.1)*, where $\{\varphi_k\}$, $\{\psi_k\}$ are orthogonal systems in the spaces H and M, respectively, and the condition* (1.6.7) *is satisfied. Then*

$$u_m = \varphi^s(I(f)) = \sum_{k=1}^{m} (\sigma_k(\psi_k,\psi_k)(\varphi_k,\varphi_k))^{-1}(f,\psi_k)\varphi_k$$

is a linear spline algorithm for the solution operator $S = (A^*A)_\infty^{-1}$ and information $I(f) = [(f,\varphi_1),\cdots,(f,\varphi_m)]$. Moreover, these approximate solutions converge to the solution of the equation (5.2.4) *in the energetic space $E_{(A^*A)_\infty}$ of the operator $(A^*A)_\infty$, and also in the space $D((A^*A)^{-\infty})$.*

5.3 Application of the obtained results for an approximate solution of a CT problem

Computerized tomography (CT) is the numerical reconstruction of functions from their linear or plane integrals or from integrals over arbitrary manifolds. CT has applications in various fields. The most famous example of the use of CT is in x-ray diagnostics. We considered the physical scheme of the process in Section 1.7.

Here we will use the well-known singular decompositions [114] of the Radon transform to construct spline algorithms for the computerized tomography problem in some Fréchet spaces, where these problems are correct. We will use the notation of Section 1.7.

Recall that the mapping of a function defined on the p-dimensional Euclidean space $\mathbb{R}^p$ to the set of its integrals over hyperplanes in $\mathbb{R}^p$ is called the Radon transform. More precisely, if S^{p-1} is the unit sphere in $\mathbb{R}^p$, $\theta \in S^{p-1}$ and $s \in \mathbb{R}^1$, then

$$\mathfrak{R}u(\theta, s) = \int\limits_{(x,\theta)=s} u(x)\,dx = \int\limits_{\theta^\perp} u(s\theta + y)\,dy, \tag{5.3.1}$$

where integration is carried out over a hyperplane perpendicular to the vector θ and located at a distance s (taking into account the sign) from the origin.

An operator defined on the Schwartz space $S(\mathbb{R}^p)$ admits continuous continuation from the space $L^2(\mathbb{R}^p)$ to $L^2(Z)$. It is known that the operator (5.3.1) admits a singular value decomposition. For $\nu = \frac{p}{2}$, this expansion has the form ([114], p. 114)

$$\mathfrak{R}u(\theta, s) = \sum_{m=0}^{\infty} {\sum_{l\le m}}' \sigma_{ml} \sum_{k=1}^{N(n,l)} (u, u_{mlk})_{L_2(\Omega^n)} f_{mlk}(\theta, s), \tag{5.3.2}$$

where in $\sum'$ the summation occurs over the values l for which $l + m$ is an even number; the sequences u_{mlk} and f_{mlk} are orthonormal bases in the spaces $L_2(\mathbb{R}^p)$ and $L_2(Z)$, respectively.

The functions u_{mlk} from (5.3.2) have the form

$$u_{mlk}(x) = q_{ml}(|x|^2)|x|^l Y_{lk}\left(\frac{x}{|x|}\right), \quad x \in \Omega^p,$$

where q_{ml} are polynomials of order m satisfying the conditions

$$\frac{1}{2}\int\limits_0^1 t^{l+\frac{n-2}{2}} q_{ml}(t) q_{kl}(t)\,dt = \begin{cases} 1, & m = l, \\ 0, & m \ne l, \end{cases}$$

and which coincide with the accuracy of normalization with the Jakobi polynomials $G_k\big(l + \frac{n-2}{2}, l + \frac{n-2}{2}, t\big)$,

$$f_{mlk}(\theta, s) = c(m) Y_{lk}(\theta) v_m(s), \quad v_m(s) = (1 - s^2)^{\frac{p-1}{2}} C_m^{\frac{p}{2}}(s),$$

$C_m^{\frac{p}{2}}$ are Gegenbauer polynomials, i.e. algebraic polynomials of m-th order, orthogonal to $[-1, 1]$ with weight $(1 - s^2)^{\frac{p-1}{2}}$ and normalized by the condition $C_m^{\frac{p}{2}}(-1) = 1$; $c(m) = \left(\int_{-1}^{1} (1 - s^2)^{\frac{p-1}{2}} v_m^2(s)\,ds\right)^{-\frac{1}{2}}$ is a normalization coefficient, $Y_{lk}(\theta)$ is linear independent spherical harmonics of l-th order, where

$k = 1, 2, \dots, N(p, l)$,

$$N(p,l) = \frac{(2l+p-2)(p+l-3)!}{l!(p-2)!}, \quad N(p,0) = 1.$$

The positive singular values σ_{ml} are given by the equality

$$\sigma_{ml}^2 = |S^{p-2}| \frac{\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p+1}{2})} \int_{-1}^{1} C_m^{\frac{p}{2}}(t) C_l^{\frac{p-2}{2}}(t)(1-t^2)^{\frac{p-3}{2}} dt,$$

$m \in \mathbb{N}_0$, $l = m, m-2, \dots$, $|S^{p-2}|$ is surface area of the unit $(p-2)$-dimensional sphere.

It is known ([114], p. 39) that in the Schwarz space $S(\mathbb{R}^n)$ the operator $\mathfrak{R}$ is injective. Since $S(\mathbb{R}^p)$ is dense in $L_2(\Omega^p)$, it is clear that $\mathfrak{R}$ is also injective in $L_2(\Omega^p)$. The above also implies the density of $\operatorname{Im}\mathfrak{R}$ in $L_2(Z)$.

Thus, a generalized solution in the Moore–Penrose sense of the equation $\mathfrak{R}u = f$ is the unique solution to the equation $\mathfrak{R}^*\mathfrak{R}u = \mathfrak{R}^* f$, which belongs to $\overline{\operatorname{Im}\mathfrak{R}^*}$, where $\mathfrak{R}^*$ is the conjugate operator of $\mathfrak{R}$ in the Hilbert sense. Taking into account (5.3.2) and the above formulas, we obtain

$$(\mathfrak{R}^*\mathfrak{R}u)(x) = \sum_{m=0}^{\infty} {\sum_{l \le m}}' \sigma_{ml}^2 \sum_{k=1}^{N(m,l)} (u, u_{mlk})_{L_2(\Omega)} u_{mlk}(x).$$

Let us define the space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$ using the equality

$$D((\mathfrak{R}^*\mathfrak{R})^{-\infty}) \ni u = (u, (\mathfrak{R}^*\mathfrak{R})^{-1}u, (\mathfrak{R}^*\mathfrak{R})^{-2}u, \dots) = \operatorname{orb}((\mathfrak{R}^*\mathfrak{R})^{-1}, u)$$

and the sequence of hilbertian norms

$$\|u\|_n = \left(\|u\|^2 + \|(\mathfrak{R}^*\mathfrak{R})^{-1}u\|^2 + \dots + \|(\mathfrak{R}^*\mathfrak{R})^{-n}u\|^2\right)^{\frac{1}{2}}, \quad n \in \mathbb{N}_0,$$

which are generated by the inner products

$$(u,v)_n = (u,v) + ((\mathfrak{R}^*\mathfrak{R})^{-1}u, (\mathfrak{R}^*\mathfrak{R})^{-1}v) + \dots + ((\mathfrak{R}^*\mathfrak{R})^{-n}u, (\mathfrak{R}^*\mathfrak{R})^{-n}v).$$

The orbital operator $(\mathfrak{R}^*\mathfrak{R})^{-\infty} : D((\mathfrak{R}^*\mathfrak{R})^{-\infty}) \to D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$ is determined by the equality

$$(\mathfrak{R}^*\mathfrak{R})^{-\infty}u = \{(\mathfrak{R}^*\mathfrak{R})^{-1}u, (\mathfrak{R}^*\mathfrak{R})^{-2}u, \dots\} = \operatorname{Orb}\left((\mathfrak{R}^*\mathfrak{R})^{-1}, (\mathfrak{R}^*\mathfrak{R})^{-1}u\right).$$

Its inverse operator $(\mathfrak{R}^*\mathfrak{R})_\infty = ((\mathfrak{R}^*\mathfrak{R})^{-\infty})^{-1}$ takes the form

$$\begin{aligned}(\mathfrak{R}^*\mathfrak{R})_\infty u &= \{(\mathfrak{R}^*\mathfrak{R})u, u, (\mathfrak{R}^*\mathfrak{R})^{-1}u, (\mathfrak{R}^*\mathfrak{R})^{-2}u, \dots\} \\ &= \operatorname{Orb}\big((\mathfrak{R}^*\mathfrak{R})^{-1}, (\mathfrak{R}^*\mathfrak{R})^{-1}u\big).\end{aligned}$$

The orbital operator $(\mathfrak{R}^*\mathfrak{R})^{-\infty}$ is self-adjoint and positive definite in the countable Hilbert space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$, it is an isomorphism of the space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$ onto itself. Therefore, its inverse operator $(\mathfrak{R}^*\mathfrak{R})_\infty$ is an isomorphism of the Fréchet space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$ onto itself and the equation

$$(\mathfrak{R}^*\mathfrak{R})_\infty u = f \tag{5.3.3}$$

has the unique stable solution.

Approximate solutions defined according to (5.1.4) have the form

$$u_m(x) = \sum_{j=0}^{m} \sum_{l\le j}{}' \sigma_{jl}^{-2} \sum_{k=1}^{N(p,l)} (\mathfrak{R}^* f, u_{mlk})_{L_2(\Omega^p)} u_{mlk}(x).$$

From this and from the formula (5.2.3) it turns out that

$$u_m(x) = \sum_{j=0}^{m} \sum_{l\le j}{}' \sigma_{jl}^{-1} \sum_{k=1}^{N(p,l)} (f, f_{jlk}) u_{jlk}(x).$$

In the case of $p = 2$ it turns out ([114], p. 115) that $\sigma_{ml}^2 = \frac{4\pi}{m+1}$ for any l, $N(2,l) = 2$, $Y_{l1}(t) = \cos\pi lt$, $Y_{l2}(t) = \sin\pi lt$, $u_m(s) = (1-s^2)^{\frac{1}{2}} C_m^1(s) = (1-s^2)^{\frac{1}{2}} \frac{\sin(n+1)\arccos s}{\sin\arccos s}$, $c(m) = \Big(\int_{-1}^{1}(1-s^2)^{\frac{3}{2}}(C_m^1(s))^2 ds\Big)^{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}$.

According to Proposition 1.7.1, we have that the system $\{v_{mlk}^\nu, u_{mlk}^\nu, \sigma_{ml}\}$, $m \ge 0$, $0 \le l \le m$, $k = 1, \dots, N(n,l)$, where v_{mlk}^ν, u_{mlk}, σ_{ml} are defined according to (1.6.14)–(1.6.16), represents a singular system for the Radon transformation $\mathfrak{R}$ acting from $L_2(\Omega^n, W_\nu^{-1})$ to $L_2(Z, w_\nu^{-1})$. In other words,

$$\mathfrak{R}u(\omega,s) = \sum_{m=0}^{\infty} \sum_{l\le m}{}' \sigma_{ml} \sum_{k=1}^{N(n,l)} (u, v_{mlk}^\nu)_{L_2(\Omega^n, W_\nu^{-1})} \cdot u_{mlk}^\nu(\omega,s),$$

where $\sum'$ means that the summation is carried out only for even $m+l$.

If v_{mlk}^ν, u_{mlk}^ν, σ_{ml}, $l \le m$, $1 \le k \le N(n,l)$, are presented according to (1.6.14)–(1.6.16), then

$$\lim_{m\to\infty} \sigma_{ml} \|u_{mlk}^\nu\| \cdot \|v_{mlk}^\nu\| = 0.$$

We can apply Theorem 5.2.1 and find that it is valid

Theorem 5.3.1. *Let $\{v^{\nu}_{rlk}, u^{\nu}_{rlk}, \sigma_{rl}\}$, $l \le r$, $1 \le k \le N(n,l)$, be a singular system for the Radon transform $\mathfrak{R}$, which acts from $L_2(\Omega^n, W_\nu^{-1})$, $\nu > n/2-1$, into the space $L_2(Z, w_\nu^{-1})$. Then the algorithm*

$$\varphi^s(I(f))(x) = \sum_{r=0}^{m} {\sum_{l\le r}}' \sigma_{rl} \sum_{k=1}^{N(n,l)} (f, u^{\nu}_{rlk})_{L_2(Z,w_\nu^{-1})} v^{\nu}_{rlk}(x),\ x \in \Omega^n, \tag{5.3.4}$$

where $\sum'$ means that the summation is carried out only for even $m+l$, is linear spline for the solution operator $S = (\mathfrak{R}^*\mathfrak{R})^{-1}_{\infty}$ and non-adaptive information $I(f) = [(f, u^{\nu}_{001}), \dots, (f, u^{\nu}_{mmN(n,m)})]$. In addition, these approximate solutions converge to the solution of the equation (5.3.3) *in the energetic space $E_{(\mathfrak{R}^*\mathfrak{R})_\infty}$, as well as in the space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$.*

We can rewrite (5.3.4) in the form

$$\varphi^s(I(f))(x) = W_\nu(x) \sum_{r=1}^{m} q_r(x),$$

where

$$q_r(x) = \sum_{l\le r} h_{rl} |x|^l P^{(\nu-n/2, l+n/2-1)}_{(r-l)/2} (2|x|^2-1) Y_{lk}(x/|x|),$$

$$h_{rl} = d_{rl}\sigma_{rl} \sum_{k=1}^{N(n,l)} (f, w_\nu(s) C^{\nu}_r(s) Y_{lk}(\omega))_{L_2(Z,w_\nu^{-1})},$$

and $\sum'$ means that summation is carried out only for even $m+l$.

We now consider the case when $H := L^2(\mathbb{R}^p, w_p)$ is the space of square-integrable functions on $\mathbb{R}^p$ with Hermite weight $w_p(x) = \pi^{p/2}\exp(|x|^2)$ with product $(f,g)_H = \int_{\mathbb{R}^p} f(x)\cdot \overline{g(x)}\, w_p(x)\,dx$. In [33], it is proved that the Radon operator $\mathfrak{R}$ is a continuous operator from H to the Hilbert space $M := L_2(\mathbb{S}^{p-1}\times \mathbb{R}, w_1)$ with inner product $(f,g)_M = \int_{\mathbb{S}^{p-1}}\int_{\mathbb{R}} f(u,s)\overline{g(u,s)}\, w_1(s)\,ds\,du$, where $\mathbb{S}^{p-1}$ is the unit sphere in $\mathbb{R}^p$, and the weight function $w_1(s)$ has the form $w_1(s) = \pi^{1/2}\exp(s^2)$. The acting from H to M operator $\mathfrak{R}$ has a singular decomposition [33]. To describe this decomposition, we give some notation: $\{Y_{lk}, k = 1,\dots,N(p,l)\}$ is orthogonal basis of spherical functions defined on $\mathbb{S}^{p-1}$, where $l \in \mathbb{N}_0$ and $N(p,l) = \frac{(2l+p-2)(p+l-3)!}{l!(p-2)!}$, $p \ge 2$; $\nu = p/2-1$; C^{ν}_l is Gegenbauer polynomial of order l and index ν; $L^{(\alpha)}_k$ denotes the k-th normalized Laguerre polynomial;

$$g^{\nu}_{mlk}(u,s) = Y_{kl}(u)\,\frac{H_m(s)}{w_1(s)}\,,\quad u\in\mathbb{S}^{p-1},\quad s\in\mathbb{R};$$

H_m is the m-th Hermite polynomial, which is normalized so that the functions g^ν_{mlk} are orthonormal in the space M;

$$\sigma^2_{lm} = \frac{|\mathbb{S}^{p-2}|}{C^\nu_l(1)} \int_{-1}^{1} t^m C^\nu_l(t)(1-t^2)^{\nu-1/2} dt;$$

$$q^\nu_{mlk}(x) = (-1)^{(m-l)/2} \sigma^2_{lm}((m-l)/2)! 2^m \frac{|x|^l}{w_p(|x|)} Y_{lk}(x/|x|) L^{(l+\nu)}_{(m-l)/2}(|x|^2).$$

In [33], it is proved that $(g^\nu_{mlk}, q^\nu_{mlk}, \sigma_{ml})$ represents a singular system for $\mathfrak{R}$.

Theorem 5.3.2. *Let H and M be the above-defined Hilbert spaces and $\mathfrak{R}$ be the Radon operator acting from H to M. Then the algorithm*

$$\varphi^s(I(f))(x) = \sum_{r=0}^{m} \sum_{l \le r}{}' \frac{1}{\sigma_{rl}} \sum_{k=1}^{N(p,l)} (f, g^\nu_{rlk})_M q^\nu_{rlk}(x)((q^\nu_{rlk}, q^\nu_{rlk})_H)^{-1}, \quad x \in \mathbb{R},$$

where $\sum'$ means that the summation is carried out only for even values of the numbers $m+l$, is a linear spline for the solution operator $S = (\mathfrak{R}^*\mathfrak{R})^{-1}_\infty$ and non-adaptive information $I(f) = [(f, g^\nu_{001}), \ldots, (f, g^\nu_{mmN(p,m)})]$. In addition, approximate solutions converge to the solution (5.3.3) *in the energetic space $E_{(\mathfrak{R}^*\mathfrak{R})_\infty}$, as well as in the space $D((\mathfrak{R}^*\mathfrak{R})^{-\infty})$.*

CHAPTER 6

Orbitization of quantum mechanics and central spline algorithms in Fréchet–Holbert spaces of all orbits

"Quantum mechanics is probably the most successful scientific theory ever invented. It has an astonishing range of applications from quarks and leptons to neutron stars and white dwarfs and the accuracy with which its underlying ideas have been tested is equally impressive. Yet, from its very inception, prominent physicists have expressed deep reservations about its conceptual foundations and leading figures continue to argue that it is incomplete in its core. Time and again, attempts have been made to extend it in a nontrivial fashion. Some of these proposals have been phenomenological, aimed at providing a 'mechanism' for the state reduction process. Thus, while there is universal agreement that quantum mechanics is an astonishingly powerful working tool, in the 'foundation of physics circles' there has also been a strong sentiment that sooner or later one would be forced to generalize it in a profound fashion."

To this aim we may be guided by the "correspondence principle" as stated by P. A. M. Dirac: "Classical Mechanics must be a limiting case of quantum mechanics. We should expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics and, from the understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well-known results in classical mechanics." By analogy of "correspondence principle": quantum mechanics is a limiting case of finite n-order orbital quantum mechanics, because when $n = 0$ we obtain quantum mechanics. We should expect to find that important concepts in quantum mechanics correspond to important concepts in finite n-order orbital quantum mechanics and, from the understanding of the general nature of the analogy between quantum and finite n-order orbital quantum mechanics, we may hope to get laws and theorems in finite n-order orbital quantum mechanics appearing as simple generalizations of well-known results in quantum mechanics. But, infinite-order orbital quantum mechanics, which uses the technique of Fréchet spaces, is essentially a generalization of quantum mechanics, where the Schrödinger equation takes on a new meaning. Due to its importance, we call this process of generalization of quantum mechanics the orbitization of quantum mechanics.

Mathematical models are often used to describe theoretical physical phenomena of quantum mechanics. The models then mathematically manipulate and analyze particle theories and hypotheses, i.e. the explanation of particle phenomena is organized through mathematical theories and theoretical models.

"The right way of creating new physics is different: one should begin with a beautiful mathematical idea. But it should be really beautiful! No special relations to physics is compulsory. But if it is really beautiful, it will certainly match useful physical applications, though it is not predefined, what sort of applications and where: it depends on physical consequences which may be extracted from the mathematical scheme" (P. A. M. Dirac).

In this process of mutual development of mathematics and theoretical physics, existing models are developed and new mathematical models of physics are created, which more adequately describe the quantum mechanical processes.

6.1 Orbits of observable operators at the wave functions, orbital spaces, orbital operators and orbital equations containing hamiltonian of quantum harmonic oscillator (QHO)

Our mathematical idea is to create finite orbits and orbits of observable selfadjoint operators position, momentum and energy at the states of quantum Hilbert space $L^2(R)$ ("quantum Hilbert space" means simply the Hilbert space associated with a given quantum system ([68], Section 13.1, p. 255)). Also the creation of Hilbert space of finite orbits and the graded Fréchet–Hilbert space of all orbits whose elements are the orbits of the observable operators at some elements of the space $L^2(R)$. The definition of the orbital operators corresponding to these observable operators in these spaces of orbits is given. For more adequate studying of the particle behavior the equations containing orbital operators are considered. The equations containing orbital operators of hamiltonian of QHO in the Hilbert space of finite orbits and in the graded Fréchet–Hilbert space of all orbits was considered in [170].

The definitions of finite orbits and of orbits was introduced, respectively, in [169] and [163]. We present the following reasoning from [163]: let H be a Hilbert space. Let A be a linear operator mapping H into itself. We will call the sequence $\operatorname{orb}(A, x) = (x, Ax, A^2x, \dots)$ the orbit of the operator A at the point x, i.e. $\mathrm{orb}(A, x)$ is an element of the space H^N. If $A^j x \in H$ only for $j = 0, 1, \ldots, n$, then we denote the finite sequence $(x, Ax, \ldots, A^n x)$ by $\mathrm{orb_n}(A, x)$ and call it the n-orbit of the operator A at the point $x \in H$, i.e. $\mathrm{orb_n}(A, x)$ is an element of the space H^{n+1}. In [145], the following concept was introduced: let X be a linear metric space and A be a linear continuous operator mapping X into itself. We will write $\vartheta(A, x) = \{A^n x; n \in \mathbb{N}_0\}$ and call $\vartheta(A, x)$ the orbit of $x \in X$ with respect to the operator A. Note that, in this case the set $\vartheta(A, x)$, is a subset of the linear metric space X.

Therefore, in [163], the notion of an orbit of A at a point is introduced, while in [145] is introduced the notion of orbit of x with respect to the operator A, i.e. these notions are different as subsets, are different as terms and notations. In [163], the continuity of the operator A is not assumed. We also consider the concepts of an orbital operator A_n (see [169]) that acts in the Hilbert spaces of finite orbits and orbital operator A^∞ that acts in the Fréchet–Hilbert spaces of all orbits [163] (see also [183]).

The orbital spaces and equations associated with the ill-posed problems were also considered in [168]. The Hilbert spaces of finite orbits and Fréchet–Hilbert space of all orbits represent generalization of the quantum Hilbert space and are obtained from this space by strengthening its topology. Note that the Fréchet space of all orbits $D(H^\infty)$ is defined as projective limit of the sequence $\{D(H^n)\}$ of finite orbital spaces. When $n = 0$, the Hilbert space of finite n-orbits coincide with the quantum Hilbert space, and the notion of the orbital operators corresponding to all above mentioned observable operators also coincides with the operators in quantum mechanics. We believe, that orbital quantum mechanics essentially improves the possibility to consider new computational processes that are not contained in the frames of Banach spaces and had not been considered up to now.

One of the axioms of quantum mechanics states, “To each real-valued function f on the classical phase space there is associated a self-adjoint operator $\widehat{f}$ on the quantum Hilbert space”. The operator $\widehat{f}$ is called the quantization of f. There is considered the quantization’s of a few very special classical observables, such as position, momentum, and energy ([68], Section 13, p. 255). For a particle moving in $\mathbb{R}$ the classical phase space is $\mathbb{R}^2$ with the pairs (x, p), where x being the particle’s position and p being its momentum. In that case if the function f is the position function, $f(x, p) = x$, then the associated operator $\widehat{f}$ is the position operator X, given by multiplication by x, i.e. quantization of position function is position operator X, defined by equality

$$X\psi(x) = x\psi(x) \tag{6.1.1}$$

If f is the momentum function $f(x, p) = p$, then $\widehat{f}$ is the momentum operator

P, defined by equality

$$P\psi(x) = -i\hbar \frac{d\psi}{dx}(x), \tag{6.1.2}$$

where $\hbar$ is the Plank's constant. Note that quantization of xp, i.e. $(xp)\hat{}$ is neither XP nor PX, they are not self-adjoint and $XP \neq PX$. In this case a reasonable candidate for the quantization would be $\widehat{xp} = \frac{1}{2}(XP + PX)$.

It is well-known that the position and momentum operators do not commute, but satisfy the relation

$$[X, P] = XP - PX = i\hbar I, \tag{6.1.3}$$

on $D([X, P]) = D(XP) \cap D(PX)$, where $[X, P]$ is the commutator and I is the identity operator on the space $L^2(\mathbb{R})$. $D(XP) = \{u \in D(P), Pu \in D(X)\})$, and likewise for $D(PX)$. This relation is known as the canonical commutation relation.

One of the important model systems in quantum mechanics is the harmonic oscillator. This is a system capable of performing harmonic oscillations. In physics, the model of a harmonic oscillator plays an important role, especially in the study of small oscillations of systems around a position of stable equilibrium. An example of such oscillations in quantum mechanics is the oscillations of atoms in solids, molecules, ets. The harmonic oscillator in quantum mechanics is the quantum analogue of the simple harmonic oscillator. However, here we consider not the forces acting on the particle, but the hamiltonian, that is total energy for a harmonic oscillator, in which there is a parabolic potential energy. For the hamiltonian $\mathcal{H}$ of the quantum harmonic oscillator the following representation is valid

$$\mathcal{H}\psi = \frac{P^2\psi}{2m} + m\,\frac{\omega^2 X^2\psi}{2} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m}{2}\omega^2 X^2\psi = aP^2\psi + bX^2, \tag{6.1.4}$$

where m is the mass of the particle, ω is the frequency of oscillator $a = \frac{1}{2m}$, $b = m\frac{\omega^2}{2}$, $P^2\psi = -\hbar^2\frac{d^2\psi(x)}{dx^2}$ and $X^2\psi = x^2\psi(x)$. According to ([68], Section 13.1), the hamiltonian $\mathcal{H}$ is quantization of classical hamiltonian $\mathcal{H}(x,p) = ap^2 + bx^2$, since each term contains only x or only p. The first term in the hamiltonian represents the kinetic energy of the particle, and the second term represents its potential energy.

The mathematical model of quantum mechanics describe quantum-mechanical systems by vectors of separable complex quantum Hilbert space ([68], Section 13.1, p. 255) H and with unbounded self-adjoint operators defined on them. The quantum Hilbert space in this case is usually Hilbert space $L^2(\mathbb{R})$, the elements of which are called states of quantum-mechanical systems. To each observable physical quantities corresponds a self-adjoint operator on H. Such classical observables are above mentioned hamiltonian $\mathcal{H}$ of the quantum harmonic oscillator, which corresponds to the observable "energy", the position operator X and momentum operator P.

In the case of particle moving in real line R, the operators $\mathcal{H}$, X and P are described by unbounded self-adjoint operators in $H = L^2(\mathbb{R}$. Neither the position nor the momentum operator are defined as mappings the entire Hilbert space $L^2(\mathbb{R})$ into itself. After all, for $\psi \in L^2(\mathbb{R})$ the function $x\psi(x)$ may fail to be in $L^2(\mathbb{R})$. Similarly, a function ψ in $L^2(\mathbb{R})$ may fail to be differentiable, and even if it is differentiable, the derivative may fail to be in $L^2(\mathbb{R})$. The operators X and P are unbounded operators in the space $L^2\mathbb{R})$.

Later, in the 50s of the XX century, the basic concepts of quantum mechanics were represented by the methods of the theory of generalized function. It is very important that in the space of generalized functions observable operators became continuous. But the application of the basic and generalized function spaces are difficult because of the non-metrizability of their topologies. In Section 2.5, the topologies of basic and generalized functions are presented as projection and inductive limits of the family of strict Fréchet–Hilbert spaces and their strong duals, which simplifies the use of these spaces.

In this situation, it became necessary to replace the quantum Hilbert space with the graded Fréchet–Hilbert spaces, and to extend there the theories of self-adjoint operators and computational methods. For this purpose, we have developed the best approximation theory in Fréchet spaces [193, 198], it was studied topological and geometrical properties of strict Fréchet–Hilbert spaces [201]. The extension of selfadjoint operators theory in Fréchet–Hilbert spaces was began in [85, 86] and continued for graded Fréchet–Hilbert spaces in [163]. It was extended the Ritz method ([163], see also [162]), the least squares method [202], the theories of spline [167] and central algorithms [168].

While strengthening the quantum Hilbert space topology for the hamiltonian $\mathcal{H}$ of QHO are obtained Hilbert spaces of finite orbits $D(\mathcal{H}^n)$, $n \in \mathbb{N}_0$. This is the space of the states on which the operator $\mathcal{H}$ acts n-times. $D\left(\mathcal{H}^n\right)$ is identified as the space of n-orbits $\mathrm{orb}_n\left(\mathcal{H},\psi\right) = (\psi, \mathcal{H}\psi, \ldots, \mathcal{H}^n\psi)$ ([170], see also [209]). In this case the particle that is in the state ψ is subjected to potential energy and the observer gives us $\mathcal{H}\psi$, which is still the state because $\mathcal{H}\psi \in D(\mathcal{H}) \subset H$. It is still instantly acted upon by the potential energy of $\mathcal{H}$ and the observer gives us the state $\mathcal{H}^2\psi$. After n-action, the particle enters the $H^n\psi$ state. Totally these states can be described by an n-orbit $\mathrm{orb}_n(\mathcal{H}, \psi) = (\psi, \mathcal{H}\psi, \mathcal{H}^2\psi, \ldots, \mathcal{H}^n\psi)$.

Continues this process infinitely we get to the infinite sequence $\mathrm{orb}\left(\mathcal{H}, \psi\right) = (\psi, \mathcal{H}\psi, \mathcal{H}^2\psi, \ldots, \mathcal{H}^n\psi, \ldots)$, which we call the orbit of operator $\mathcal{H}$ at the point ψ [170].

The unbounded self-adjoint operator $\mathcal{H}$ form self-adjoint orbital operators $\mathcal{H}_n$:

$D\left(\mathcal{H}_n\right) \subset \left(L^2\left(\mathbb{R}\right)\right)^{n+1} \to \operatorname{Im} \mathcal{H}_n \subset \left(L^2(\mathbb{R})\right)^{n+1}$ defined by equality

$$\mathcal{H}_0(\psi) = \psi, \quad \mathcal{H}_n\left(\operatorname{orb}_n\left(\mathcal{H},\psi\right)\right) = \operatorname{orb}_n\left(\mathcal{H},\mathcal{H}\psi\right), \quad n \geq 1\,.$$

It is defined also the self-adjoint orbital operator $\mathcal{H}^\infty:\ D\left(\mathcal{H}^\infty\right) \to D\left(\mathcal{H}^\infty\right)$ by equality

$$\mathcal{H}^\infty \operatorname{orb}(\mathcal{H},\psi) = \operatorname{orb}(\mathcal{H},\mathcal{H}\psi),$$

i.e. action of orbital operator $\mathcal{H}^\infty$ on $\operatorname{orb}(\mathcal{H},\psi)$ means the action $\mathcal{H}$ on all coordinates of the orbit in the space of all orbits $D(\mathcal{H}^\infty)$. $D\left(\mathcal{H}^\infty\right)$ is a well-known space and after the introduction of orbital operator $\mathcal{H}^\infty$, the space $D\left(\mathcal{H}^\infty\right)$ acquired new content. This Fréchet–Hilbert space of all orbits coincides to the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$. The significance of this space for quantum mechanics is also denoted in [13]. $D(\mathcal{H}^\infty)$ is the projective limit of the sequence of spaces $\{D(\mathcal{H}^n)\}$, i.e. the study of computational processes in space $D\left(\mathcal{H}^\infty\right)$ can be reduced to the study of computational processes in space $D(\mathcal{H}^n)$ [170]. In computational mathematics problems, this means that the equation given in the Fréchet–Hilbert space $D\left(\mathcal{H}^\infty\right)$ is projected onto the $D\left(\mathcal{H}^n\right)$ spaces and calculation of the ε-complexity in the Fréchet space of all orbits is reduced to calculate the ε-complexity in some n-orbit Hilbert space. Note that the self-adjoint operator $\mathcal{H}^\infty$ is topological isomorphism onto the space $D\left(\mathcal{H}^\infty\right)$. That is, the flaw of von Neumann's theory was somewhat corrected. This orbital operator $\mathcal{H}^\infty$ has also recently appeared in the paper [210].

The equation $\mathcal{H}u = f$ containing the operator $\mathcal{H}$, which in the space $D(\mathcal{H}^n)$ (resp. in the space $D(\mathcal{H}^\infty)$) has the form $\mathcal{H}_n(\operatorname{orb}_n\left(\mathcal{H},u\right)) = \operatorname{orb}_n\left(\mathcal{H},f\right)$ (resp. $\mathcal{H}^\infty \operatorname{orb}(\mathcal{H},u) = \operatorname{orb}(\mathcal{H},f)$), is considered. For the obtained equations, a linear spline central algorithm is constructed in the Hilbert space $D\left(\mathcal{H}^n\right)$ (resp. in the Fréchet space $D(\mathcal{H}^\infty)$) [170]. Construction of spline algorithms for the ill-posed problem of computerized tomography in the spaces of orbits $D(R^*R)^{-n})$ and $D(R^*R)^{-\infty})$, where R is Radon transform, is given in [208] and [168]. Similarly, the ε-complexity will be calculated for the computerised tomography problem in the Hilbert space of finite orbits and spline algorithms built in the Fréchet space of all orbits. Analogously are defined the spaces $D(X^n)$ and $D(X^\infty)$, operators X_n and X^∞ for the position operator X. As well are defined the spaces $D(P^n)$ and $D(P^\infty)$, operators P_n and P^∞ for the momentum operator P [210].

Generalization of *canonical commutation relations* between X_n and P_n in the space of orbits has the following form

$$X_n P_n \operatorname{orb}_n(X,\psi) - P_n X_n \ \operatorname{orb}_n(X,\psi) = i\hbar \operatorname{orb}_n(X,\psi)$$

and is given in [210]. In this paper, the generalization of Heisenberg uncertainly principle for orbital operators is also given. The norms of orbital spaces $D(X^n)$

and $D(P^n)$ are strengthening the topology of the space $L^2(\mathbb{R})$. The creation of orbits of operators, orbital spaces, orbital operators, we call *orbitization* and the results obtained *orbital quantum mechanics*.

While orbitization of the quantum Hilbert space the states of the quantum-mechanical systems is replaced by the space of the orbits of the operators at the states. The self-adjoint operators in the quantum Hilbert space are replaced by an orbital self-adjoint operator in the Hilbert space of finite orbits and in the Fréchet–Hilbert space of all orbits, allowing for a more adequate description of the observable quantities. Orbitization also considers creation of equations containing orbital operators in the corresponding orbital spaces [209]. The properties of operators is changed as a result of orbitisation. In particular, the unbounded quantum harmonic oscillator operator becomes a topological isomorphism in the Fréchet–Hilbert spaces of all orbits, in this case it coincides with the Schwartz space of rapidly decreasing functions. The work [13] is dedicated to the special importance of this space for quantum mechanics.

Thus, the represented orbits $\mathrm{orb}_n\,(\mathcal{H},\psi)$, $\mathrm{orb}(\mathcal{H},\psi)$ and the orbital operators $\mathcal{H}_n$ and $\mathcal{H}^\infty$ more adequately describe the state of the particle because we have the whole infinite sequence of observer data on the particle. For the required modeling accuracy, the study of computational processes associated with an infinite sequence of observer data is reduced to the study of computational processes with a finite data sequence. This was considered in [170], for calculation of the inverse of the harmonic oscillator in the spaces of orbits and in [168] for computerized tomography problem. This process is coordinated by a functional (quasinorm of metric) built specifically by us in Section 2.4. That is, it is a matter of bringing an infinite coordinate computational process to a finite coordinate computational process based on certain requirements or other considerations for accuracy. Orbital quantum mechanics will similarly study orbits, orbital operators, orbital spaces and orbital equations for position and momentum observables X and P. As well as for operators of creation C, annihilation A and numerical N. Each of the considered operators produce n-finite orbits $\mathrm{orb}_n(\mathcal{H},\psi)$, $\mathrm{orb}_n(X,\psi)$, $\mathrm{orb}_n(P,\psi)$, $\mathrm{orb}_n(C,\psi)$, $\mathrm{orb}_n(A,\psi)$, $\mathrm{orb}_n(N,\psi)$ $(n \in \mathbb{N}_0)$ and orbits $\mathrm{orb}(\mathcal{H},\psi)$, $\mathrm{orb}(X,\psi)$, $\mathrm{orb}(P,\psi)$, $\mathrm{orb}(C,\psi)$, $\mathrm{orb}(A,\psi)$, $\mathrm{orb}(N,\psi)$ in the state ψ of quantum Hilbert space. They also generate n-finite orbital operators $\mathcal{H}_n$, X_n, P_n, C_n, A_n, N_n, which act accordingly on the Hilbert space of finite n-orbits $D(\mathcal{H}_n)$, $D(X_n)$, $D(P_n)$, $D(C_n)$, $D(A_n)$, $D(N_n)$. These operators also generate orbital operators $\mathcal{H}^\infty$, X^∞, P^∞, C^∞, A^∞, N^∞ that operate accordingly , $D(\mathcal{H}^\infty)$, $D(X^\infty)$, $D(P^\infty)$, $D(C^\infty)$, $D(A^\infty)$, $D(N^\infty$ in the Fréchet space of all orbits. The quantization of classical physics and our creation of the basis for finite n-order orbital quantum mechanics and infinite-order orbital quantum mechanics are schematically given in the following table (when $n = 0$, a classical case is obtained).

Classical phase space $\mathbb{R}^2$. Pairs (x,p) with x being the particle's position and p being its momentum	Q u a n t i z a t i o n		O r b i t i z a t i o n of observable operators in the case of Hilbert space of n-orbits		O r b i t i z a t i o n of observable operators in the case of Fréchet–Hilbert space of all orbits	
	The quantum Hilbert space $L^2(\mathbb{R}^2)$	Observable operators obtained from Quantization of classical function	The quantum Hilbert space of n-orbits	The finite orbital operators corresponding to observable operators	The quantum Fréchet–Hilbert space of all orbits	orbital operators corresponding to observable operators
The position function, $f(x,p)=x$	The quantum Hilbert space $L^2(\mathbb{R}^2)$	The associated operator $\hat{f}$ is the self-adjoint position operator $X\psi(x)=x\psi(x)$	The Hilbert space of finite orbits $D(X_n)=D(X^n)\subset(L^2(\mathbb{R}^2))^n$ $(n\in\mathbb{N})$	The finite orbital operator $X_n: D(X^n)\to D(X^n)$, $X_n\,\mathrm{orb}_n(X,\psi)=\mathrm{orb}_n(X,X\psi)$ corresponding to X $(n\in\mathbb{N})$	The quantum Fréchet–Hilbert space of all orbits $D\,(X^\infty)$	The orbital operator $X^\infty: D(X^\infty)\to D(X^\infty)$, $X^\infty\mathrm{orb}(X,\psi)=\mathrm{orb}(X,X\psi)$ corresponding to operator X
The momentum function $f(x,p)=p$	The quantum Hilbert space $L^2(\mathbb{R}^1)$	The momentum self-adjoint operator $P\psi=-ihd\psi/dx$	The Hilbert space of finite orbits $D(P_n)=D(P^n)\subset(L^2(\mathbb{R}^1))^n$ $(n\in\mathbb{N})$	The finite orbital operator $P_n: D(P^n)\to D(P^n)$, $P_n\,\mathrm{orb}_n(P,\psi)=\mathrm{orb}_n(P,P\psi)$ corresponding to P $(n\in\mathbb{N})$	The quantum Fréchet–Hilbert space of all orbits $D(P^\infty)$	The orbital operator $P^\infty: D(P^\infty)\to D(P^\infty)$, $P^\infty\,\mathrm{orb}(P,\psi)=\mathrm{orb}(P,P\psi)$ corresponding to operator P
The energy classical hamiltonian $H(x,p)=ap^2+bx^2$	The quantum Hilbert space $L^2(\mathbb{R}^1)$	The selfadjoint quantum Harmonic oscillator (QHO) $H\psi=aP^2\psi+bX^2\psi$	The Hilbert space of finite orbits $D(H_n)=D(H^n)\subset(L^2(\mathbb{R}^1))^n$ $(n\in\mathbb{N})$	The finite orbital operator $H_n: D(H^n)\to D(H^n)$, $H_n\,\mathrm{orb}_n(H,\psi)=\mathrm{orb}_n(H,H\psi)$ corresponding to QHO H $(n\in\mathbb{N})$	The quantum Fréchet–Hilbert space of all orbits $D(H^\infty)$	The orbital operator $H^\infty: D(H^\infty)\to D(H^\infty)$, $H^\infty\,\mathrm{orb}(H,\psi)=\mathrm{orb}(H,H\psi)$ corresponding to operator H
The creation operator C	The quantum Hilbert space $L^2(\mathbb{R}^1)$	The creation operator $C\psi=-ihd\psi/dx+x/2$	Hilbert space of finite orbits $D(C_n)=D(C^n)\subset(L^2(\mathbb{R}^1))^n$ $(n\in\mathbb{N})$	The finite orbital operator $C_n\,\mathrm{orb}_n(C,\psi)=\mathrm{orb}_n(C,C\psi)$ corresponding to C $(n\in\mathbb{N})$	The quantum Fréchet–Hilbert space of all orbits $D(C^\infty)$	The orbital operator $C^\infty: D(C^\infty)\to D(C^\infty)$, $C^\infty\,\mathrm{orb}(C,\psi)=\mathrm{orb}(C,C\psi)$ corresponding to operator C
The annihilation operator A	The quantum Hilbert space $L^2(\mathbb{R}^1)$	The annihilation operator $A\psi=ihd\psi/dx+x/2$	Hilbert space of orbits $D(A_n)=D(A^n)\subset(L^2(\mathbb{R}^1))^n$ $(n\in\mathbb{N})$	The orbital operator $A_n\,\mathrm{orb}_n(A,\psi)=\mathrm{orb}_n(A,A\psi)$ corresponding to A $(n\in\mathbb{N})$	The quantum Fréchet–Hilbert space of all orbits $D(A^\infty)$	The orbital operator $A^\infty: D(A^\infty)\to D(A^\infty)$, $A^\infty\,\mathrm{orb}(A,\psi)=\mathrm{orb}(A,A\psi)$ corresponding to operator A

In classical physics, all observables commute	$\psi \in D(X) \cap D(P)$	Canonical commutation relation $(XP - PX)\psi = ih\psi$	The state of quantum system $\psi_n = (\psi_1, \ldots, \psi_n) \in D(X^n) \cap D(P^n)$	Canonical commutational relation $(X_n P_n - P_n X_n)\psi_n = ih\psi_n$	The state of quantum system $\psi = (\psi_1, \ldots, \psi_n, \ldots) \in D(X^\infty) \cap D(P^\infty)$	Canonical commutational relation $(X^\infty P^\infty - P^\infty X^\infty)\psi = ih\psi$
Equation in phase space $H(x,p) = ap^2 + bx^2$	The quantum Hilbert space $L^2(\mathbb{R}^1)$	The equation $\mathcal{H}u = f$	The n-orbital Hilbert space $D(H^n)$	Orbital equation $H_n \operatorname{orb}_n(H,\psi) = \operatorname{orb}_n(H,f)$, constructed linear spline central algorithm	The orbital Fréchet–Hilbert space $D(H^\infty)$	Orbital equation $H^\infty \operatorname{orb}(H,\psi) = \operatorname{orb}_n(H^\infty, f)$, constructed linear spline central algorithm

This new mathematical model - orbital quantum mechanics essentially improved the possibilities of computations and gives possibility to consider new computational processes that not contained in the frames of Hilbert spaces and was not considered up to now.

6.2 A generalization of the canonical commutation relation and Heisenberg Uncertainty Principle for the orbital operators

6.2.1 Finite orbits of operators at the states, orbital operators corresponding to the position and the momentum operators in the Hilbert space of finite n-orbits

Let $n \in \mathbb{N}_0$ and consider elements of the space $L^2(\mathbb{R})$, to which the power of position operator $X^n = X(X^{n-1})$ (X^0 is the identical operator) can be applied. The space of such elements is denoted by $D(X^n)$, besides $D(X^0) = L^2(\mathbb{R})$. It is well known that for all $\psi \in D(X)$, the quantity $\langle \psi, X\psi \rangle$ is real. More generally, if all of $\psi, X\psi, \ldots, X^n\psi$ belong to $D(X)$, then $\langle \psi, X^n\psi \rangle$ is real. By n-orbits of the operator X at the point $\psi \in L^2(\mathbb{R})$ we mean a finite sequence

$$\operatorname{orb}_n(X, \psi) := (\psi, X\psi, \ldots, X^n\psi) = (\psi, x\psi, \ldots, x^n\psi). \tag{6.2.1}$$

The space $D(X^n)$ we identify with the space of n-orbirs of the operator X. For the injective operator, each of the orbits $\operatorname{orb}_n(X, \psi)$ is uniquely determined by the element $\psi \in L^2(\mathbb{R})$, which we call the generated element of orbit (6.2.1).

We define now the orbital operator

$$X_n : D(X_n) \subset (L^2(\mathbb{R}))^{n+1} \to \operatorname{Im} X_n \subset (L^2(\mathbb{R}))^{n+1}$$

corresponding to the position operator (6.1.2) defined by the equality

$$X_0(\psi) = \psi,\ X_n(\operatorname{orb}_n(X, \psi)) = \operatorname{orb}_n(X, X\psi),\ n \in \mathbb{N}. \tag{6.2.2}$$

According to (6.2.2) the following representation is valid

$$X_n(\psi, X\psi, \ldots, X^n\psi) = (X\psi, X^2\psi, \ldots, X^{n+1}\psi)$$
$$= (x\psi, x^2\psi, \ldots, x^{n+1}\psi) = x\mathrm{orb}_n(X, \psi);$$

also $\psi, X\psi, \ldots, X^n\psi$, belong to the domain of definition of the operator X. The orbital operator X_n can be extended to $D(X)^{n+1} \subset (L^2(\mathbb{R}))^{n+1}$ by means of the following equality

$$X_n(\psi_0, \psi_1, \ldots, \psi_n) = (X\psi_0, X\psi_1, \ldots, X\psi_n) = x(\psi_0, \psi_1, \ldots, \psi_n).$$

We can turn $D(X^n)$ into a Hilbert space using the following inner product

$$\langle \mathrm{orb}_n(X, \varphi), \mathrm{orb}_n(X, \psi)\rangle_n$$
$$= (\varphi, \psi) + (X\varphi, X\psi) + \cdots + (X^n\varphi, X^n\psi),\ n \in \mathbb{N}_0, \tag{6.2.3}$$

and with the norm corresponding to (6.2.3)

$$\|\mathrm{orb}_n(X, \psi)\|_n = \langle (\psi, \psi) + (X\psi, X\psi) + \cdots + (X^n\psi, X^n\psi)\rangle_n^{1/2}$$
$$= (\|\psi\|^2 + \|X\psi\|^2 + \|X^2\psi\|^2 + \cdots + \|X^n\psi\|^2)^{1/2}, \tag{6.2.4}$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ are the inner product and the norm of $L^2(\mathbb{R})$. It is easy to see that the operator X_n is a linear unbounded symmetric operator in the Hilbert space $D(X^n)$ with a dense image. In ([68], Section 9, Corollary 9.1), it is proved also that the position operator X is self-adjoint in $L^2(\mathbb{R})$. Therefore, the orbital operator X_n has an analogous property in the space $D(X_n)$.

According to the standard definitions of the probability theory, the expected value of the position is

$$E(X) = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx,$$

provided that the integral is absolutely convergent ([68], Section 3, formula (3.2)). More generally, we can compute any moment of the position i.e. the expected value of the power m of the position as ([68], Section 3, formula (3.3))

$$E(X^m) = \int_{-\infty}^{\infty} x^m|\psi(x)|^2 dx, \quad m \in \mathbb{N},$$

assuming again the convergence of the integral. A key idea in quantum theory is to express the expected values of various quantities (position, momentum, energy, etc.) in terms of operators and the inner product in the Hilbert space $L^2(\mathbb{R})$. In the case of position operator this expression of expected value has the following form

$$E(X^m) = \langle \psi(x), X^m \psi(x) \rangle,\ m \in \mathbb{N}.$$

In this case norm (6.2.4) of the space $D(X^n)$ is expressed by the integral as follows

$$\begin{aligned}
\|\mathrm{orb}_n(X,\psi)\|_n^2 &= (\psi,\psi) + (\psi, X^2\psi) + \cdots + (\psi, X^{2n}\psi) \\
&= (\psi,\psi) + E(X^2) + \cdots + E(X^{2m}) \\
&= \int\limits_{-\infty}^{\infty} |\psi(x)|^2 dx + \int\limits_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx + \cdots + \int\limits_{-\infty}^{\infty} x^{2m} |\psi(x)|^2 dx \\
&= \int\limits_{-\infty}^{\infty} |\psi(x)|^2 (1 + x^2 + \cdots + x^{2m}) dx.
\end{aligned}$$

By analogy to (6.2.1) we define

$$\begin{aligned}
\mathrm{orb}_n(P,\psi) &:= (\psi, P\psi, \ldots, P^n\psi) \\
&= (\psi, (-i\hbar)d\psi/dx, \ldots, (-i\hbar)^n d^n\psi/dx^n),\ n \in \mathbb{N}_0,
\end{aligned} \tag{6.2.5}$$

as well as the space $D(P^n)$.

Now let us define the orbital operator

$$P_n : D(P_n) \subset (L^2(\mathbb{R}))^{n+1} \to \mathrm{Im} P_n \subset (L^2(\mathbb{R}))^{n+1}$$

corresponding to the momentum operators (6.1.2) defined by the equality

$$P_0(\psi) = \psi,\ P_n(\mathrm{orb}_n(P,\psi)) = \mathrm{orb}_n(P, P\psi),\ n \geq 1. \tag{6.2.6}$$

According to (6.2.5) the following representation holds

$$\begin{aligned}
P_n(\psi, P\psi, \ldots, P^n\psi) &= ((-i\hbar)d/dx)_n \mathrm{orb}_n(P,\psi) \\
&= (-i\hbar)d/dx)_n (\psi, P\psi, \ldots, P^n\psi).
\end{aligned}$$

We assume that $\psi, P\psi, \ldots, P^n\psi$ belong to the domain of definition of the operator P. The orbital operator $P_n = ((-i\hbar)d/dx)_n$ corresponding to the momentum operator P can be extended to $D(P)^{n+1} \subset (L^2(\mathbb{R}))^{n+1}$ as follows

$$\begin{aligned}
P_n(\psi_0, \psi_1, \ldots, \psi_n) &= (P\psi_0, P\psi_1, \ldots, P\psi_n) \\
&= ((-i\hbar)d\psi_0/dx, (-i\hbar)d\psi_1/dx, \ldots, (-i\hbar)d\psi_n/dx).
\end{aligned}$$

Indeed, if $\mathrm{orb}_n(X,\psi)\in D(P^n)$, then $P_n\mathrm{orb}_n(X,\psi)=(P\psi, PX\psi,\dots,PX^n\psi)$.

We can turn $D(P^n)$ into a Hilbert space using the following inner product

$$\langle \mathrm{orb}_n(P,\varphi),\mathrm{orb}_n(P,\psi)\rangle_n=(\varphi,\psi)+(P\varphi,P\psi)+\cdots+(P^n\varphi,P^n\psi),\ n\in\mathbb{N}_0,$$

and with the corresponding norm

$$\begin{aligned}\|\mathrm{orb}_n(P,\psi)\|_n&=((\psi,\psi)+(P\psi,P\psi)+\cdots+(P^n\psi,P^n\psi))^{1/2}\\&=(\|\psi\|^2+\|P\psi\|^2+\|P^2\psi\|^2+\cdots+\|P^n\psi\|^2)^{1/2}.\end{aligned}\tag{6.2.7}$$

Due to ([68], Proposition 3, equality (3.13)) for the norm (6.2.7) the following representation holds:

$$\begin{gathered}\|\mathrm{orb}_n(P,\psi)\|_n^2=(\psi,\psi)+(\psi,P^2\psi)+\cdots+(\psi,P^{2n}\psi)\\=\int_{-\infty}^{\infty}|\widehat{\psi}(k)|^2dk+\int_{-\infty}^{\infty}|\hbar k|^2|\widehat{\psi}(k)|^2dk+\cdots+\int_{-\infty}^{\infty}|\hbar k|^{2n}|\widehat{\psi}(x)|^2dk\\=\int_{-\infty}^{\infty}|\widehat{\psi}(k)|^2(1+|\hbar k|^2+\cdots+|\bar{k}h|^{2n})dk,\end{gathered}$$

where under the integral $\widehat{\psi}(k)$ we understand the Fourier transform of the function ψ. $(\psi,P^m\psi)$ is interpreted as the expected value of the m-th power of the momentum $E(P^m)$. It is known that the momentum operator P is essentially self-adjoint in $L^2(\mathbb{R})$ ([68], Section 9, Proposition 9.29). Therefore, the operator P_n has the analogous property in the Hilbert space $D(P^n)$.

Remark 6.2.1. It was noted that the function $|\psi(x)|^2$ is the probability density for the position of the particle. This means that the probability that the particle belongs to some set $E\subset R$ is $\int_E|\psi(x)|^2dx$ ([68], Section 3.3, p. 58). For this prescription to make sense, ψ should be normalized so that $\int_R|\psi(x)|^2dx=1$. The probability that the particle belongs to some set $E\subset R$ must be the same in the space $D(\mathcal{H}^n)$ and analogously should be equal to

$$\begin{gathered}\langle \mathrm{orb}_n(\mathcal{H},\psi),\mathrm{orb}_n(\mathcal{H},\psi)\rangle_{n,E}/\|\mathrm{orb}_n(\mathcal{H},\psi)\|_n^2\\=\Big(\int_E|\psi(x)|^2dx+\int_E|\mathcal{H}\psi(x)|^2dx+\cdots+\int_E|\mathcal{H}^n\psi(x)|^2dx\Big)\Big/\|\mathrm{orb}_n(\mathcal{H},\psi)\|_n^2\le 1.\end{gathered}$$

The wave function $\psi_k(x)$ is a solution to the Schrodinger equation

$$\mathcal{H}\psi(x)=E_k\psi(x),$$

where $\mathcal{H}$ is defined by equality (6.1.4), $E_k = \hbar\omega(k+1/2),\ k \in \mathbb{N}_0$, and

$$\psi_k(x) = (-1)^k (2^k k! \sqrt{\pi})^{-1/2} d^k e^{-x^2}/dx^k, \quad k \in \mathbb{N}_0.$$

We also note that

$$\mathcal{H}_n \mathrm{orb}_n(\mathcal{H}, \psi_k) = E_k \mathrm{orb}_n(\mathcal{H}, \psi_k), \quad n, k \in \mathbb{N}_0.$$

It turns out that for the functions ψ_k

$$\begin{aligned} &\langle \mathrm{orb}_n(\mathcal{H}, \psi_k), \mathrm{orb}_n(\mathcal{H}, \psi_k)\rangle_{n,E} / \|\mathrm{orb}_n(\mathcal{H}, \psi_k)\|_n^2 \\ &= \Big((1 + E_k^2 + \cdots + E_k^{2n}) \int\limits_E |\psi_k(x)|^2 dx\Big) \Big/ \|\psi_k\|^2 (1 + E_k^2 + \cdots + E_k^{2n}) \\ &= \int\limits_E |\psi_k(x)|^2 dx, \quad k, n \in \mathbb{N}_0. \end{aligned}$$

This means that, in the considered case, the probability of location of a particle in the set E will again be $\int_E |\psi_k(x)|^2 dx$.

6.2.2 Generalized canonical commutation relation for orbital operators corresponding to the momentum and position operators in the Hilbert space of n-finite orbits

We prove now the generalized canonical commutation relations between X_n and P_n that in the case $n = 0$ coincides with equality (6.1.3).

Theorem 6.2.1. *For the commutator $[X_n, P_n] = X_n P_n - P_n X_n$ the following statements hold true:*

a) *If $(\psi_0, \psi_1, \ldots, \psi_n) \in D([X_n, P_n]) = D(X_n P_n) \cap D(P_n X_n)$, then*

$$[X_n, P_n](\psi_0, \psi_1, \ldots, \psi_n) = i\hbar(\psi_0, \psi_1, \ldots, \psi_n).$$

b) *If $\mathrm{orb}_n(P, \psi) \in D(X_n P_n)$ and $\mathrm{orb}_n(X, \psi) \in D(P_n X_n)$, then*

$$\begin{aligned} &X_n P_n \mathrm{orb}_n(P, \psi) - P_n X_n \mathrm{orb}_n(X, \psi) \\ &= (i\hbar I\psi, XP^2\psi - PX^2\psi, \ldots, XP^{n+1}\psi - PX^{n+1}\psi). \end{aligned}$$

c) *If $\mathrm{orb}_n(X, \psi) \in D(X_n P_n)$ and $\mathrm{orb}_n(P, \psi) \in D(P_n X_n)$, then*

$$\begin{aligned} &X_n P_n \mathrm{orb}_n(X, \psi) - P_n X_n \mathrm{orb}_n(P, \psi) \\ &= (i\hbar I\psi, XPX\psi - PXP\psi, \ldots, XPX^n\psi - PXP^n\psi). \end{aligned}$$

Proof. a) $[X_n, P_n](\psi_0, \psi_1, \dots, \psi_n) = X_n P_n(\psi_0, \psi_1, \dots, \psi_n)$

$$-P_n X_n(\psi_0, \psi_1, \dots, \psi_n)$$

$$= X_n(P\psi_0, P\psi_1, \dots, P\psi_n) - P_n(X\psi_0, X\psi_1, \dots, X\psi_n)$$
$$= (XP\psi_0, XP\psi_1, \dots, XP\psi_n) - (PX\psi_0, PX\psi_1, \dots, PX\psi_n)$$
$$= ((XP - PX)\psi_0, (XP - PX)\psi_1, \dots, (XP - PX)\psi_n)$$
$$= (i\hbar\psi_0, i\hbar\psi_1, \dots, i\hbar\psi_n) = i\hbar(\psi_0, \psi_1, \dots, \psi_n).$$

b) $X_n P_n \mathrm{orb}_n(P, \psi) - P_n X_n \mathrm{orb}_n(X, \psi) = X_n \mathrm{orb}_n(P, P\psi) - P_n \mathrm{orb}_n(X, X\psi)$

$$= (XP\psi, XP^2\psi, \dots, XP^{n+1}\psi) - (PX\psi, PX^2\psi, \dots, PX^{n+1}\psi)$$

$$= (XP\psi - PX\psi, XP^2\psi - PX^2\psi, \dots, XP^{n+1}\psi - PX^{n+1}\psi).$$

c) $X_n P_n \mathrm{orb}_n(X, \psi) - P_n X_n \mathrm{orb}_n(P, \psi)$

$$= X_n P_n(\psi, X\psi, \dots, X^n\psi) - P_n X_n(\psi, P\psi, \dots, P^n\psi)$$

$$= X_n(P\psi, PX\psi, \dots, PX^n\psi) - P_n(X\psi, XP\psi, \dots, XP^n\psi)$$
$$= (XP\psi, XPX\psi, \dots, XPX^n\psi) - (PX\psi, PXP\psi, \dots, PXP^n\psi)$$
$$= (XP\psi - PX\psi, XPX\psi - PXP\psi, \dots, XPX^n\psi - PXP^n\psi). \qquad \square$$

Corollary. *Statement a) of Theorem* 6.2.1 *implies*

1. *If* $\mathrm{orb}_n(X, \psi) \in D([X_n, P_n]) = D(X_n P_n) \cap D(P_n X_n)$*, then*

$$[X_n, P_n]\mathrm{orb}_n(X, \psi) = i\hbar \mathrm{orb}_n(X, \psi).$$

2. *If* $\mathrm{orb}_n(P, \psi) \in D([X_n, P_n]) = D(X_n P_n) \cap D(P_n X_n)$*, then*

$$[X_n, P_n]\mathrm{orb}_n(P, \psi) = i\hbar \mathrm{orb}_n(P, \psi).$$

If $n = 0$, in all cases for the first coordinate we obtain the following canonical commutation relation

$$[X_n, P_n]\psi = [X, P]\psi = XP\psi - PX\psi = i\hbar I\psi.$$

Now we investigate the orbitization of the operator $\mathcal{H}$ and established relationship between the orbital operators $\mathcal{H}_n, X_n$ and P_n.

Theorem 6.2.2. *The following statements are valid:*

a) *Let $B : D(B) \subset H \to H$ and $C : D(C) \subset H \to H$ be the linear operators in a Hilbert space H and $(B+C)_n$ be the n-orbital operator, corresponding to the operator $B+C$. Then for the orbital operators B_n and C_n, corresponding to B and C, the equality*

$$(B+C)_n(\psi_0, \psi_1, \ldots, \psi_n) = B_n(\psi_0, \psi_1, \ldots, \psi_n) + C_n(\psi_0, \psi_1, \ldots, \psi_n)$$

holds for $(\psi_0, \psi_1, \ldots, \psi_n) \in D(B_n) \cap D(C_n)$. In particular, $(B+C)_n \mathrm{orb_n}(B, \psi) = B_n \mathrm{orb_n}(B, \psi) + C_n \mathrm{orb_n}(B, \psi)$, but $\mathrm{orb}_n(B+C, \psi) \neq \mathrm{orb}_n(B, \psi) + \mathrm{orb}_n(C, \psi)$.

For the orbital operator $(BC)_n$, corresponding to BC, the equality

$$(BC)_n(\psi_0, \psi_1, \ldots, \psi_n) = B_n C_n(\psi_0, \psi_1, \ldots, \psi_n)$$

holds for $(\psi_0, \psi_1, \ldots, \psi_n) \in D(B_n) \cap D(C_n)$. In particular, $(BC)_n \mathrm{orb_n}(B, \psi) = B_n C_n \mathrm{orb_n}(B, \psi)$ and $(BC)_n \mathrm{orb_n}(C, \psi) = B_n C_n \mathrm{orb_n}(C, \psi)$.

b) *For the operator $\mathcal{H}_n = (aP^2 + bX^2)_n$, corresponding to the hamiltonian $\mathcal{H}$, the following equalities are valid:*

$$\begin{aligned} \mathcal{H}_n(\psi_0, \psi_1, \ldots, \psi_n) &= (aP^2 + bX^2)_n(\psi_0, \psi_1, \ldots, \psi_n) \\ &= (aP_n^2 + bX_n^2)(\psi_0, \psi_1, \ldots, \psi_n) \\ &= aP_n^2(\psi_0, \psi_1, \ldots, \psi_n) + bX_n^2(\psi_0, \psi_1, \ldots, \psi_n). \end{aligned}$$

c) $\mathcal{H}_n \mathrm{orb_n}(\mathcal{H}, \psi) = ((aP^2+bX^2)\psi, (aP^2+bX^2)^2\psi, \ldots, (aP^2+bX^2)^{n+1}\psi)$.

d) $\mathcal{H}_n \mathrm{orb_n}(\mathcal{H}, \psi) = ((aP_n^2 \mathrm{orb_n}(aP^2 + bX^2, \psi) + bX_n^2 \mathrm{orb_n}(aP^2 + bX^2, \psi)$.

e) *If $\mathrm{orb}_n(P, \psi) \in D(X_n P_n)$ and $\mathrm{orb}_n(X, \psi) \in D(P_n X_n)$, then*

$$\begin{aligned} &X_n P_n \mathrm{orb}_n(P, \psi) + P_n X_n \mathrm{orb}_n(X, \psi) \\ &\qquad = (XP\psi + PX\psi, 2PXP\psi, \ldots, XP^{n+1}\psi + PX^{n+1}\psi). \end{aligned}$$

Proof. a) $(B+C)_n(\psi_0, \psi_1, \ldots, \psi_n) = ((B+C)\psi_0, (B+C)\psi_1, \ldots, (B+C)\psi_n) = (B\psi_0 + C\psi_0, B\psi_1 + C\psi_1, \ldots, B\psi_n + C\psi_n) = (B\psi_0, B\psi_1, \ldots, B\psi_n) + (C\psi_0, C\psi_1, \ldots, C\psi_n) = B_n(\psi_0, \psi_1, \ldots, \psi_n) + C_n(\psi_0, \psi_1, \ldots, \psi_n)$.

$$\begin{aligned} &(BC)_n(\psi_0, \psi_1, \ldots, \psi_n) = (BC\psi_0, BC\psi_1, \ldots, BC\psi_n) \\ &= B_n(C\psi_0, C\psi_1, \ldots, C\psi_n) = B_n C_n(\psi_0, \psi_1, \ldots, \psi_n). \end{aligned}$$

b) $\mathcal{H}_n(\psi_0, \psi_1, \ldots, \psi_n) = (aP^2 + bX^2)_n(\psi_0, \psi_1, \ldots, \psi_n)$. It follows from a) that $(aP^2 + bX^2)_n(\psi_0, \psi_1, \ldots, \psi_n) = (aP_n^2 + bX_n^2)(\psi_0, \psi_1, \ldots, \psi_n) = aP_n^2(\psi_0, \psi_1, \ldots, \psi_n) + bX_n^2(\psi_0, \psi_1, \ldots, \psi_n)$.

c) $\mathcal{H}_n\mathrm{orb_n}(\mathcal{H},\psi) = \mathcal{H}_n\mathrm{orb_n}(aP^2+bX^2,\psi) = ((aP^2+bX^2)\psi,(aP^2+bX^2)^2\psi,\dots,(aP^2+bX^2)^{n+1}\psi)$.

d) $\mathcal{H}_n\mathrm{orb_n}(\mathcal{H},\psi) = (aP^2+bX^2)_n\mathrm{orb_n}(\mathcal{H},\psi) = ((aP^2+bX^2)\psi,(aP^2+bX^2)\mathcal{H}\psi,\dots,(aP^2+bX^2)\mathcal{H}^n\psi) = ((aP^2\psi+bX^2\psi),(aP^2\mathcal{H}\psi+bX^2\mathcal{H}\psi),\dots,(aP^2\mathcal{H}^n\psi+bX^2\mathcal{H}^n\psi) = (aP^2\psi,aP^2\mathcal{H}\psi,\dots,aP^2\mathcal{H}^n\psi)+(bX^2\psi,bX^2\mathcal{H}\psi,\dots,bX^2\mathcal{H}^n\psi) = aP_n^2\mathrm{orb_n}(\mathcal{H},\psi)+bX_n^2\mathrm{orb_n}(\mathcal{H},\psi) = aP_n^2\mathrm{orb_n}(aP^2+bX^2,\psi)+bX_n^2\mathrm{orb_n}(aP^2+bX^2,\psi)$.

e) We obtain the equality $X_nP_n\mathrm{orb}_n(P,\psi)+P_nX_n\mathrm{orb}_n(X,\psi) = (XP\psi+XP\psi,XP^2\psi+PX^2\psi,\dots,XP^{n+1}\psi+PX^{n+1}\psi)$. According to ([68], Section 1.13, formula (13.4)), we obtain that $XP^2\psi+PX^2\psi = 2PXP\psi$ and this can be expressed as the Weil quantization of $f(x,p)=xp^2$, i.e. the following relation holds:

$$Q_{weil}(xp^2) = 1/3(XP^2\psi+PX^2\psi+PXP\psi) = 1/2(XP^2\psi+PX^2\psi) = PXP.$$

Therefore, we get

$$\begin{aligned}&X_nP_n\mathrm{orb}_n(P,\psi)+P_nX_n\mathrm{orb}_n(X,\psi)\\&=(XP\psi+XP\psi,2PXP\psi,\dots,XP^{n+1}+PX^{n+1}\psi).\end{aligned}$$

In comparing different quantization schemes it is important to see that two different expressions may describe the same operator. □

The operators P and X are secondary commuting, i.e. $[[P,X],P]=0$ and $[[P,X],X]=0$. If pairs of operators satisfy canonical commutation relation, then they are secondary commuting.

Proposition 6.2.3. *For the orbital operators X_n and P_n, corresponding to X and P, we have*

$$[[X_n,P_n],P_n]\equiv 0,\quad [[X_n,P_n],X_n]\equiv 0.$$

6.2.3 The Fréchet space of all orbits and a generalization of the canonical commutation relations

Note that for a general positive definite operator A with a discrete spectrum (like the hamiltonian $\mathcal{H}$ is), the space $D(A^\infty)$ circumstantially was studied in ([160], Chapter 8), where $D(A^\infty)$ was the whole symbol and A^∞, did not have sense if taken separately. According to ([138], Section X.6), $D(A^\infty)$ is the set of all infinitely differentialle elements of A and is denoted by $C^\infty(A)$. In [163], we have defined the operator A^∞ as follows

$$A^\infty(\psi,A\psi,\dots,A^n\psi,\dots) = (A\psi,A^2\psi,\dots,A^{n+1}\psi,\dots), \tag{6.2.8}$$

or $A^\infty \mathrm{orb}(A, \psi) = \mathrm{orb}(A, A\psi)$.

Due to this notation, the space $D(A^\infty)$ acquare new meaning that differs from the classical case. Namely, now $D(A^\infty)$ is also the domain of definition of the operator A^∞, defined by equality (6.2.8).

We denote by $D(X^\infty)$ the intersection $\cap_{n=0}^\infty D(X^n)$ of the spaces $D(X^n)$. This means that we can apply the operator X to a function from $D(X^\infty)$ infinitely many times. So that $f \in D(X^\infty)$ if and only if $f \in L^2(\mathbb{R})$ together with the its products by arbitrary polynomials.

The space $D(X^\infty)$ is isomorphic to the space of all orbits $\mathrm{orb}(X, \psi) = \{\psi, X\psi, \dots, X^n\psi, \dots\}$ of the operator X at the states ψ and this isomorphism is obtained by the mapping $D(X^\infty) \ni \psi \to \mathrm{orb}(X, \psi)$.

It is easy to prove that the space $D(X^n) \supset D(X^\infty) \supset D(H^\infty)$, where H is the hamiltonian of quantum harmonic oscillator. $D(H^\infty)$ is isomorphic to the Schwartz space of rapidly decreasing functions [170] and is a nonempty set of second category.

The topology of the space $D(X^\infty)$ is generated by the sequence of norms (6.2.4). The space $D(X^\infty)$ is also the domain of definition of the operator X^∞ defined by

$$\begin{aligned} X^\infty(\psi(x), X\psi(x), \dots, X^{n-1}\psi(x), \dots) \\ = D(X^\infty(\mathrm{orb}(X, \psi)) = x\mathrm{orb}(X, \psi). \end{aligned} \tag{6.2.9}$$

It will be also noted below that the space $D(X^\infty)$ can be represented as a projective limit of a sequence of the Hilbert spaces $\{D(X^n)\}$.

Problem 6.2.1. It is not known whether the metrizable LCS $D(X^\infty)$ is nuclear and countable-Hilbert.

In the case of a momentum operator P the space of all orbits $D(P^\infty)$ with the sequence of norms (6.2.7) is defined analogously. The space $D(P^\infty)$ is also the domain of definition of the operator P^∞ defined by the equality

$$P^\infty(\psi, P\psi, \dots, P^n\psi, \dots) = (P\psi, P^2\psi, \dots, P^{n+1}\psi, \dots). \tag{6.2.10}$$

This means that $P^\infty(\psi, P\psi, \dots, P^n\psi, \dots) = ((-i\hbar)d/dx)^\infty \mathrm{orb}(P, \psi)$, where the operator P^∞ is indeed defined by the equality

$$\begin{aligned} P^\infty \mathrm{orb}(P, \psi) = ((-i\hbar)d\psi/dx)^\infty \mathrm{orb}(P, \psi) \\ = ((-i\hbar)d\psi/dx, (-i\hbar)^2 d^2\psi/dx^2, \dots, (-i\hbar)^{n+1} d^{n+1}\psi/dx^{n+1}, \dots), \end{aligned}$$

According to ([68], p. 569, Theorem 10.7.5), the operator P as an operator in $L^2(\mathbb{R})$ is self-adjoint. Therefore, according to ([163], Theorem 3, statement b), the operator P^∞ defined by equality (6.2.10) is self-adjoint in the Fréchet space $D(P^\infty)$.

Problem 6.2.2. It is not known whether the Fréchet space $D(P^\infty)$ is nuclear and countable-Hilbert.

Theorem 6.2.4. *For the commutator*

$$[X^\infty, P^\infty] = X^\infty P^\infty - P^\infty X^\infty,$$

where X^∞ is defined bythe equality (6.2.9) *and P^∞ is defined by equality* (6.2.10)*, the following relations are taking place:*

a) *If* $(\psi_0, \dots, \psi_n, \dots) \in D([X^\infty, P^\infty]) = D(X^\infty P^\infty) \cap D(P^\infty X^\infty)$*, then*

$$[X^\infty, P^\infty](\psi_0, \dots, \psi_n, \dots) = i\hbar(\psi_0, \dots, \psi_n, \dots).$$

b) *If* $\operatorname{orb}(P, \psi) \in D(X^\infty P^\infty)$ *and* $\operatorname{orb}(X, \psi) \in D(P^\infty X^\infty)$*, then*

$$\begin{gathered} X^\infty P^\infty \operatorname{orb}(P, \psi) - P^\infty X^\infty \operatorname{orb}(X, \psi) \\ = i\hbar(I\psi, XP^2\psi - PX^2\psi, \dots, XP^{n+1}\psi - PX^{n+1}\psi, \dots). \end{gathered}$$

c) *If* $\operatorname{orb}(X, \psi) \in D(X^\infty P^\infty)$ *and* $\operatorname{orb}(P, \psi) \in D(P^\infty X^\infty)$*, then*

$$\begin{gathered} X^\infty P^\infty \operatorname{orb}(X, \psi) - P^\infty X^\infty \operatorname{orb}(P, \psi) \\ = i\hbar(I\psi, XPX\psi - PXP\psi, \dots, XPX^n\psi - PXP^n\psi, \dots). \end{gathered}$$

The statement a) gives us a direct generalization of the canonical commutation relation. The statements b) and c) also represent a generalization of the canonical commutation relation.

Corollary. *It follows from the statement a) of Theorem 6.2.4 that*

1. *If* $\operatorname{orb}(X, \psi) \in D([X^\infty, P^\infty]) = D(X^\infty P^\infty) \cap D(P^\infty X^\infty)$*, then*

$$X^\infty P^\infty \operatorname{orb}(X, \psi) - P^\infty X^\infty \operatorname{orb}(X, \psi) = i\hbar \operatorname{orb}(X, \psi).$$

2. *If* $\operatorname{orb}(P, \psi) \in D([X^\infty, P^\infty]) = D(X^\infty P^\infty) \cap D(P^\infty X^\infty)$*, then*

$$X^\infty P^\infty \operatorname{orb}(P, \psi) - P^\infty X^\infty \operatorname{orb}(P, \psi) = i\hbar \operatorname{orb}(P, \psi).$$

Now we investigate the orbitization of the hamiltonian operator $H\psi = aP^2\psi + bX^2\psi$ and established relationship between the orbital operators H^∞, X^∞ and P^∞.

Proposition 6.2.5. *The following representations are valid:*

a) *For the orbital operator $\mathcal{H}^\infty = (aP^2+bX^2)^\infty$, corresponding to the hamiltonian $\mathcal{H}$, the following relation holds true*

$$(\mathcal{H}^\infty)^2(\psi_0,\dots,\psi_n,\dots) = (\mathcal{H}^2)^\infty(\psi_0,\dots,\psi_n,\dots).$$

b) *For the orbital operator $\mathcal{H}^\infty = (aP^2+bX^2)^\infty$, the following relation holds true*

$$\begin{aligned}\mathcal{H}^\infty(\psi_0,\dots,\psi_n,\dots) &= (aP^2+bX^2)^\infty(\psi_0,\dots,\psi_n,\dots)\\ &= (aP^2+bX^2)^\infty(\psi_0,\dots,\psi_n,\dots)\\ &= (aP^2)^\infty(\psi_0,\dots,\psi_n,\dots) + (bX^2)^\infty(\psi_0,\dots,\psi_n,\dots).\end{aligned}$$

c) *Further,* $\mathcal{H}^\infty\mathrm{orb}(\mathcal{H},\psi) = \mathcal{H}^\infty\mathrm{orb}(aP^2+bX^2,\psi) = ((aP^2+bX^2)\psi,(aP^2+bX^2)^2\psi,(aP^2+bX^2)^3\psi+\cdots,(aP^2+bX^2)^{n+1}\psi+\cdots)$.

d) *Finally,* $\mathcal{H}^\infty\mathrm{orb}(\mathcal{H},\psi) = (aP^2+bX^2)^\infty\mathrm{orb}(\mathcal{H},\psi) = (aP^2)^\infty\mathrm{orb}(\mathcal{H},\psi)+(bX^2)^\infty\mathrm{orb}(\mathcal{H},\psi))$.

Proof. a) We have

$$\begin{aligned}(\mathcal{H}^\infty)^2(\psi_0,\dots,\psi_n,\dots) &= \mathcal{H}^\infty(\mathcal{H}\psi_0,\dots,\mathcal{H}\psi_n,\dots) = (\mathcal{H}^2\psi_0,\dots,\mathcal{H}^2\psi_n,\dots)\\ &= (\mathcal{H}^2)^\infty(\psi_0,\dots,\psi_n,\dots).\end{aligned}$$

The statements b), c) and d) are proved analogously to the statements b), c) and d) of Theorem 6.2.2. □

Remark 6.2.2. The probability that the particle belongs to a set $E\subset\mathbb{R}$ calculated for the space $D(\mathcal{H}^\infty)$ for the wave function ψ, is determined by the limit

$$\lim_{n\to\infty}\langle\mathrm{orb}_n(\mathcal{H},\psi),\mathrm{orb}_n(\mathcal{H},\psi)\rangle_{n,E}/\|\mathrm{orb}_\mathrm{n}(\mathcal{H},\psi)\|_\mathrm{n}^2.$$

This limit exists, since the sequence increases and is bounded from above by the number 1. The limit $\lim\limits_{n\to\infty}\langle\mathrm{orb}_n(\mathcal{H},\psi_k),\mathrm{orb}_n(\mathcal{H},\psi_k)\rangle_{n,E}/\|\mathrm{orb}_\mathrm{n}(\mathcal{H},\psi_\mathrm{k})\|_\mathrm{n}^2$ again exists and is equal to $\int|\psi_k(x)|dx$ because it is the same for all $n,k\in\mathbb{N}_0$.

6.2.4 Generalization of Heisenberg uncertainty principle

Let $\Psi = (\psi_0, \psi_1, \ldots, \psi_n)$, where $\psi_j \in L^2(\mathbb{R})$, $j = 0, 1, \ldots, n$. If $\Psi = (\psi_0, \psi_1, \ldots, \psi_n)$ and $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_n)$ belong to $(L^2(\mathbb{R}))^{n+1}$, then their inner product $\langle \Psi, \Phi \rangle_n$ is defined by the equality

$$\langle \Psi, \Phi \rangle_n = \sum_{j=0}^{n} (\psi_j, \varphi_j),\ n \in \mathbb{N}_0,$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R})$. For an operator V acting in $L^2(\mathbb{R})$ we consider, as above, the operator $V_n(\Psi) = (V\psi_0, V\psi_1, \ldots, V\psi_n)$ acting on $(L^2(\mathbb{R}))^{n+1}$.

Let us define the mean value of V_n in the space $(L^2(\mathbb{R}))^{n+1}$ as

$$\mu := \mu_\Psi(V_n) = \langle V_n \Psi, \Psi \rangle_n.$$

For the linear selfadjoint operators S and T acting in $L^2(\mathbb{R})$ we have

$$\langle T_n \Psi, \Psi \rangle_n = \sum_{j=0}^{n} (T\psi_j, \psi_j) = \sum_{j=0}^{n} (\psi_j, T\psi_j) = \langle \Psi, T_n \Psi \rangle_n.$$

Thus T_n is selfadjoint on $(L^2(\mathbb{R}))^{n+1}$ and therefore $\mu_\Psi(T_n)$ is a real number. The same can be said on the value $\mu_\Psi(S_n)$. Now consider the value

$$\mathrm{var}_\Psi(T_n) = \langle (T - \mu I)_n^2 \Psi, \Psi \rangle_n = \sum_{j=0}^{n} \int_{\mathbb{R}} (T - \mu I)^2 \psi_j(q) \psi_j(q) dq, \quad (6.2.11)$$

where I is identical operator in $L^2(\mathbb{R})$. Since μ is real and T_n is selfadjoint, $\mathrm{var}_\Psi(T_n)$ is a nonnegative real number and we can introduce the standard deviation of T_n

$$\mathrm{sd}_\Psi(T_n) = \sqrt{\mathrm{var}_\Psi(T_n)}. \quad (6.2.12)$$

The value $\mathrm{sd}_\Psi(S_n)$ is defined analogously.

Theorem 6.2.6 (Heisenberg uncertainty principle)**.** *Let S, T be linear selfadjoint operators in the space $L^2(\mathbb{R})$ and moreover S_n and T_n are defined on $(L^2(\mathbb{R}))^{n+1}$. Then the operator $C_n = [S_n, T_n]$ satisfies the equality*

$$|\mu_\Psi(C_n)| \leq 2\mathrm{sd}_\Psi(S_n)\mathrm{sd}_\Psi(T_n). \quad (6.2.13)$$

Proof. Let us define

$$\mu_1 = \mu_\Psi(S_n),\ \mu_2 = \mu_\Psi(T_n)$$

and

$$A_n = S_n - \mu_1 I_n,\ B_n = T_n - \mu_2 I_n,\ I_n = Id_{(L^2(\mathbb{R}))^{n+1}}.$$

Since μ_1 and μ_2 are real numbers, A_n and B_n are selfadjoint operators on $(L^2(\mathbb{R}))^{n+1}$. It is easy to verify that $C_n = [S_n, T_n] = [A_n, B_n]$ and according to the statements d) and e) of Theorem 6.2.2, we obtain

$$\begin{aligned}\mu_\Psi(C_n) &= \langle (AB - BA)_n\Psi, \Psi\rangle_n = \langle A_nB_n\Psi, \Psi\rangle_n - \langle B_nA_n\Psi, \Psi\rangle_n \\ &= \langle B_n\Psi, A_n\Psi\rangle_n - \langle A_n\Psi, B_n\Psi\rangle_n. \end{aligned} \tag{6.2.14}$$

We obtain from here that

$$\begin{aligned}|\mu_\Psi(C_n)| &= 2|\mathrm{Im}\langle B_n\Psi, A_n\Psi\rangle_n| \leq 2|\langle B_n\Psi, A_n\Psi\rangle_n| \\ &\leq 2\|A_n\Psi\|_n \cdot \|B_n\Psi\|_n \end{aligned} \tag{6.2.15}$$

(here $\|\cdot\|_n$ denotes the norm in $(L^2(\mathbb{R}))^{n+1}$). Further we have

$$\begin{aligned}\|B_n\Psi\|_n &= \|(T_n - \mu_2 I_n)\Psi\|_n = (\langle (T-\mu_2 I)_n^2\Psi, \Psi\rangle_n)^{1/2} \\ &= \sqrt{\mathrm{var}_\Psi(T_n)} = \mathrm{sd}_\Psi(T_n).\end{aligned}$$

Analogously we obtain that $\|A_n\Psi\|_n = \mathrm{sd}_\Psi(S_n)$ and Theorem 6.2.6 is proved. □

If the conditions of Theorem 6.2.6 are satisfied, then in inequality (6.2.13) the equality sign is attained if and only if the two inequalities in (6.2.15) turn into equalities. This can happen in the following three cases: 1. $B_n\Psi = 0$; 2. $A_n\Psi = 0$; 3. $B_n\Psi = cA_n\Psi$ for some constant c. If $B_n\Psi = 0$, then $T_n\Psi - \mu_2\Psi = 0$, i.e. then Ψ is an eigenvector for T_n with the eigenvalue μ_2. This means that $\psi_j,\ j = 0, 1, \ldots, n$, are eigenvectors for the operator T, i.e. $T\psi_0 = \mu_2\psi_0, \ldots, T\psi_n = \mu_2\psi_n$. This condition will be valid, for example, if $\Psi = \mathrm{orb}_n(T, \psi)$, where ψ is the eigenvector of T corresponding to μ_2 (in such a case the relation $T_n(\mathrm{orb}_n(T, \psi)) = \mu_2\mathrm{orb}_n(T, \psi)$ will be valid). A similar conclusion is obtained while considering the second case: $A_n\Psi = 0$ if and only if $\psi_j,\ j = 0, 1, \ldots, n$ are eigenvectors for the operator S. This condition will be valid, for example, if $\Psi = \mathrm{orb}_n(S, \psi)$, where ψ is an eigenvector of S corresponding to the eigenvalue μ_1. Now we consider the third case, when $B_n\Psi = cA_n\Psi$ for some constant c. According to (6.2.15), $|\mu_\Psi(C_n)| = 2|\mathrm{Im}c\langle A_n\Psi, A_n\Psi\rangle_n| = 2|c|\|\langle A_n\Psi, A_n\Psi\rangle_n|$. Thus $c = i\gamma, \gamma \in \mathbb{R}$, i.e.

$$B_n\Psi = i\gamma A_n\Psi.$$

Thus, we have

$$(T_n - \mu_1 I_n)\Psi = i\gamma(S_n - \mu_2 I_n)\Psi$$

or

$$(T_n - i\gamma S_n)\Psi = (\mu 2 - i\gamma\mu_1)\Psi,$$

i.e. Ψ is an eigenvector of $T_n - i\gamma S_n$. Conversely, if Ψ is an eigenvector for $T_n - i\gamma S_n$ with some eigenvalue $\lambda = c + id$ in $\mathbb{C}$, without loss of generality we assume that $\|\Psi\|_n = 1$. Then

$$(c - id)\|\Psi\|_n^2 = \langle \Psi, (T_n - i\gamma S_n)\Psi\rangle_n = \langle \Psi, T_n\Psi\rangle_n + \langle i\gamma\Psi, S_n\Psi\rangle_n.$$

We have from here that $c = \langle \Psi, T_n\Psi\rangle_n = \mu_2$ and $d = -\gamma\langle \Psi, S_n\Psi\rangle_n = -\gamma\mu_1$. Therefore, (6.2.14) and (6.2.15) hold, and in (6.2.13) the equality takes plase). We conclude that the equality in (6.2.13) is attained, for example, if $\Psi = \mathrm{orb}_n(T - i\gamma S, \psi)$, where ψ is an eigenvalue of $T - i\gamma S$.

It follows from Theorem 6.2.6 that for the position operator X and the momentum operator P the following is true

$$\mathrm{sd}_\Psi(X_n)\mathrm{sd}_\Psi(P_n) \geq \frac{\hbar}{2}\|\Psi\|_n^2. \tag{6.2.16}$$

In Theorem 6.2.6 the uncertainty principle is proved for every Ψ, belonging to the domain $D(S_nT_n) \cap D(T_nS_n)$. Now we will see that if S_n and T_n are taken to be the usual position and momentum orbital operators X_n and P_n, the uncertainty principle is valed if Ψ belongs to both $D(X_n)$ and $D(P_n)$.

Theorem 6.2.7. *Suppose that $\Psi = (\psi_0, \psi_1, \ldots, \psi_n)$ belongs to $D(X_n) \cap D(P_n)$. Then inequality* (6.2.16) *holds.*

Proof. First let us prove that

$$\langle X_n\Psi, P_n\Psi\rangle_n = \langle P_n\Psi, X_n\Psi\rangle_n - i\hbar\langle \Psi, \Psi\rangle_n. \tag{6.2.17}$$

We have

$$\begin{aligned}\langle X_n\Psi, P_n\Psi\rangle_n &= \langle (X\psi_0, X\psi_1, \ldots, X\psi_n), (P\psi_0, P\psi_1, \ldots, P\psi_n)\rangle_n \\ &= \sum_{j=0}^{n} (X\psi_j, P\psi_j); \qquad (6.2.18)\\ (X\psi_j, P\psi_j) &= \lim_{a\to 0}\Big(X\psi_j(x), -i\hbar\frac{\psi_j(x+a) - \psi_j(x)}{a}\Big) \\ &= \lim_{a\to 0}\Big(\frac{i\hbar}{a}(x\psi_j(x), \psi_j(x+a)) - \frac{i\hbar}{a}(X\psi_j(x), \psi_j(x)\Big).\end{aligned}$$

In the first inner product we make a substitution $x = y - a$, and in the second we replace x by y. Then

$$(X\psi_j, P\psi_j) = \lim_{a\to 0}\Big(\frac{i\hbar}{a}((y-a)\psi_j(y-a), \psi_j(y)) - \frac{i\hbar}{a}(X\psi_j(y), \psi_j(y)\Big)$$
$$= \lim_{a\to 0}\Big(\frac{i\hbar}{a}(y\psi_j(y-a), \psi_j(y)) - \frac{i\hbar}{a}(y\psi_j(y), \psi_j(y)) - i\hbar(\psi_j(y-a), \psi_j(y))\Big)$$
$$= \lim_{a\to 0}\Big(-i\hbar(\frac{\psi_j(y-a) - \psi_j(y)}{-a}, y\psi_j(y)) - i\hbar(\psi_j(x), \psi_j(x))\Big)$$
$$= (P\psi_j, X\psi_j) - i\hbar(\psi_j(x), \psi_j(x)).$$

Therefore, (6.2.18) implies (6.2.17).

From (6.2.17) we obtain for arbitrary real α and β that

$$\langle (X_n - \alpha I_n)\Psi, (P_n - \beta I_n)\Psi\rangle_n = \langle (P_n - \beta I_n)\Psi, (X_n - \alpha I_n)\Psi\rangle_n - i\hbar\langle \Psi, \Psi\rangle_n.$$

Therefore,

$$\begin{aligned}\langle \Psi, \Psi\rangle_n &= \frac{1}{i\hbar}\langle (P_n - \beta I_n)\Psi, (X_n - \alpha I_n\Psi)\rangle_n \\ &\quad - \frac{1}{i\hbar}\langle (X_n - \alpha I_n)\Psi, (P_n - \beta I_n)\Psi)\rangle_n \\ &\le \frac{2}{\hbar}\|(X_n - \alpha I_n)\Psi)\|_n \cdot \|(P_n - \beta I_n)\Psi\|_n,\end{aligned} \tag{6.2.19}$$

where I_n denotes identical operator in $(L^2(\mathbb{R}))^{n+1}$. If we substitute here $\alpha = \langle X_n\Psi, \Psi\rangle_n$ and $\beta = \langle P_n\Psi, \Psi\rangle_n$, by (6.2.11) and (6.2.12), respectively then we get

$$|\langle \Psi, \Psi\rangle_n| \le \frac{2}{\hbar}\mathrm{sd}_\Psi(X_n) \cdot \mathrm{sd}_\Psi(P_n).$$

Theorem 6.2.7 is proved. □

If the conditions of Theorem 6.2.7 are satisfied, then in inequality (6.2.16), the equality sign is attained if and only if in the last line of (6.2.19) we have the equality. The equality will hold in that line if and only if one of $(X_n - \alpha I)\Psi$ and $(P_n - \beta I)\Psi$ is zero or $(P_n - \beta I)\Psi$ is a pure-imaginary multiple of $(X_n - \alpha I)\Psi$. Since we assume that Ψ is non zero element in $(L^2(\mathbb{R}))^{n+1}$, the first case is impossible. Thus we must consider only the condition

$$(X_n - \alpha I)\Psi = i\gamma(P_n - \beta I)\Psi, \tag{6.2.20}$$

where γ is a nonzero real number, $\alpha = \langle \Psi, X_n\Psi\rangle_n$ and $\beta = \langle \Psi, P_n\Psi\rangle_n$. Similar to the reasoning carried out in the analysis of Theorem 6.2.6, for our case it turns out that (6.2.20) is equivalent to the assertion that Ψ is an eigenvector for the operator $X_n - i\gamma P_n$ for some nonzero real number γ. The equation (6.2.20) is satisfied for the vector $\Psi(x) \in (L^2(\mathbb{R}))^{n+1}$, all of whose coordinates $\psi(x)$ are of the form

$$\psi(x) = C\exp\Big(\frac{x^2}{\gamma\hbar} - \frac{\alpha}{\gamma\hbar} + \frac{i\beta}{\hbar}\Big), \quad C \in \mathbb{R}, \quad \gamma < 0.$$

For example, we can take a function $\psi(x) = C\exp(x^2/(\gamma\hbar)),\ \gamma < 0, C \in \mathbb{R}$, for which $\alpha = \beta = 0$.

Theorems 6.2.6 and 6.2.7 in the case $n = 0$ were proved, respectively, in ([88], p. 579–580) and in ([68], p. 246–248).

6.3 Generalization of canonical commutational relations for orbital operators of Creation and Annihilation. Twice commutability of these operators

Introduced by Dirac creation and annihilation operators have widespread applications in quantum mechanics, notably in the study of quantum harmonic oscillators and many-particle systems. Modern quantum physics almost unthinkable without them. We create finite orbits and orbits of creation, annihilation and numerical operators at the states of quantum Hilbert space $L^2(\mathbb{R})$. The Hilbert space of finite orbits and the Fréchet–Hilbert space of all orbits which elements are the orbits of these operators at some elements of the space $L^2(\mathbb{R})$ are definite and studied. Moreover, the notion of orbital operators corresponding to these operators in the spaces of orbits is introduced and studied. We establish well-known canonical commutation relations for orbital operators corresponding to the creation and annihilation operators in the Hilbert space of finite orbits and in the Fréchet–Hilbert space of all orbits.

In Section 6.3.1, finite orbits of the creation operator C and of the annihilation operator A at the states, as well n-orbital operators C_n and A_n corresponding to creation and annihilation operators in the Hilbert space of finite orbits are defined. According to the definition of orbital operators C_n and A_n it is naturally to determinate its value on the element $(\varphi_0, \varphi_1 \ldots, \varphi_n) \in (D(C))^{n+1} \cap (D(A))^{n+1}$. We need this while proving of canonical commmutation relations between C_n and A_n because we must also consider the value C_n on the orbits of the operator A and the value A_n on the orbits of the operator C at some states. Some relations between orbital operators N_n corresponding to numerical operator N and with the operators C_n and A_n are also established. The generalized canonical commutation relations between C_n and A_n are proved that in the case $n = 0$ coincides with the classical one.

In Section 6.3.2, orbits of creation and annihilation operators at states, the Fréchet–Hilbert spaces of all orbits $D(C^\infty)$ and $D(A^\infty)$, the orbital operators C^∞ and A^∞ in these spaces are studied and generalized canonical commutation relation is proved. The analogous relationship between orbital operator $N^\infty,\ C^\infty$ and A^∞ is established.

6.3.1 Orbital operators corresponding to the creation and annihilation operators in the Hilbert space of finite orbits

A creation operator is differential operator that have the following form ([13], p.541)

$$C = -d/dx + x/2. \tag{6.3.1}$$

An annihilation operator is usually denoted by ([13], p.541)

$$A = d/dx + x/2. \tag{6.3.2}$$

Note that under the names of creation and annihilation operators, the lightly modified operators $\frac{1}{\sqrt{2}}(d/dx+x)$ and $\frac{1}{\sqrt{2}}(-d/dx+x)$ are often considered and denoted, respectively, by a^* and a ([68], Section 11.4). As well, they are denoted by $A^\dagger$ and A ([138], ch. V). They are often also denoted by $\hat{a}^\dagger$ and $\hat{a}$, or by a^+ and a. The annihilation operator thus defined reduces the number of particles in a given state by one, and the creation operator increases this number by one. Neither the creation nor the annihilation operator are defined as mappings on the entire Hilbert space $L^2(\mathbb{R})$ into itself. After all, for $\varphi \in L^2(\mathbb{R})$ the functions $C\varphi$ and $A\varphi$ may fail to be in $L^2(\mathbb{R})$. By definition, domain of definition $D(C)$ of the operator C consists of all $\psi \in L^2(\mathbb{R})$ such that $C\psi \in L^2(\mathbb{R})$. The operators C and A are unbounded operators in $L^2(\mathbb{R})$.

It is well-known that the creation and the annihilation operators do not commute, but satisfy the relation

$$[A, C] = AC - CA = I \tag{6.3.3}$$

on $D([A, C]) = D(AC) \cap D(CA),\ D(CA) = \{u \in D(A), A(u) \in D(C)\}$ and likewise for $D(AC)$. In (6.3.3) $[A, C]$ is the commutator and I is identity operator on the space $L^2(\mathbb{R})$. Really

$$AC = x^2/4 - d^2/dx^2 + 1/2I,\ CA = x^2/4 - d^2/dx^2 - 1/2I,\ AC - CA = I.$$

The relation (6.3.3) is known as the canonical commutation relation.

n-orbits of the annihilation and creation operators (6.3.1) and (6.3.2) in the state φ are defined as

$$\begin{aligned}\mathrm{orb}_n(A,\varphi) &= (\varphi, A\varphi, A^2\varphi, \ldots, A^n\varphi)\\ &= (\varphi, (d/dx + x/2)\varphi, \ldots, (d/dx + x/2)^n\varphi),\end{aligned}$$

and

$$\begin{gathered}\mathrm{orb}_n(C,\varphi) = (\varphi, C\varphi, C^2\varphi, \ldots, C^n\varphi)\\ = (\varphi, (-d/dx + x/2)\varphi, \ldots, (-d/dx + x/2)^n\varphi).\end{gathered} \tag{6.3.4}$$

It is well known ([13], formula (56)) that

$$C\psi_j = \sqrt{j+1}\psi_{j+1}, \tag{6.3.5}$$

where

$$\psi_j(x) = (-1)^j(2\pi)^{-1/4}(j!)^{-1/2}\exp(x^2/4)d^j\exp(-x^2/2)/dx^j,\ j\in\mathbb{N}_0, \tag{6.3.6}$$

are wave functions of harmonic oscillator.

For the orbit of creation operator (6.3.1) in the state ψ_j we have

$$\begin{gathered}\mathrm{orb}_n(C,\psi_j) = \{\psi_j, C\psi_j, \ldots, C^n\psi_j\}\\ = (\psi_j, \sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \ldots, \sqrt{j+1}\ldots\sqrt{j+n}\psi_{j+n})\end{gathered}$$

and

$$\begin{gathered}C_n\mathrm{orb}_n(C,\psi_j) = (C\psi_j, C^2\psi_j, \ldots, C^{n+1}\psi_j)\\ = (\sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \ldots, \sqrt{j+1}\ldots\sqrt{j+n+1}\psi_{j+n+1}).\end{gathered}$$

It is well-known([13], formula (53)), that

$$A\psi_j = \sqrt{j}\psi_{j-1}. \tag{6.3.7}$$

Therefore

$$\begin{gathered}\mathrm{orb}_n(A,\psi_j) = (\psi_j, A\psi_j, A^2\psi_j, \ldots, A^n\psi_j)\\ = (\psi_j, \sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \ldots, \sqrt{j}\sqrt{j-1}\ldots\sqrt{j-n+1}\psi_{j-n})\end{gathered}$$

and

$$\begin{gathered}A_n\mathrm{orb}_n(A,\psi_j) = (A\psi_j, A^2\psi_j, \ldots, A^{n+1}\psi_j)\\ = (\sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \ldots, \sqrt{j}\sqrt{j-1}\ldots\sqrt{j+n}\psi_{j-n-1}).\end{gathered}$$

We have

$$AC\psi_j = A(\sqrt{j+1}\psi_{j+1}) = \sqrt{j+1}A\psi_{j+1} = (j+1)\psi_j.$$

The operator

$$N = CA = x^2/4 - d^2/dx^2 - 1/2,$$

is called the number operator. We have that

$$N\psi_j = CA\psi_j = C(\sqrt{j}\psi_{j-1}) = \sqrt{j}C\psi_{j-1} = j\psi_j$$

and

$$N_n(\varphi_0, \dots, \varphi_n) = (N\varphi_0, \dots, N\varphi_n) \text{ for } (\varphi_0, \dots, \varphi_n) \in D(N_n) = (D(N))^{n+1}.$$

Theorem 6.3.1. *The following representation are valid:*

a) *If* $(\varphi_0, \varphi_1, \dots, \varphi_n) \in D(N_n)$*, then*

$$N_n(\varphi_0, \varphi_1, \dots, \varphi_n) = C_n A_n(\varphi_0, \varphi_1, \dots, \varphi_n).$$

b) *For the functions* ψ_j*, defined by formula* (6.3.6)*, we have that*

$$N_n \mathrm{orb}_n(A, \psi_j) = \big(j\psi_j, (j-1)\sqrt{j}\psi_{j-1}, (j-2)\sqrt{j}\sqrt{j-1}\psi_{j-2}, \dots,$$
$$(j-n)\sqrt{j}\sqrt{j-1}\dots\sqrt{j-n+1}\psi_{j-n}\big),\ j \in \mathbb{N}_0,$$
$$\psi_{j-n} = 0, \text{if } j < n.$$

c) *For the functions* ψ_j*, defined by formula* (6.3.6)*, we have*

$$N_n \mathrm{orb}_n(C, \psi_j) = \big(j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}(j+2)\psi_{j+2}, \dots,$$
$$\sqrt{j+1}\sqrt{j+2}\dots\sqrt{j+n}(j+n)\psi_{j+n}\big).$$

d) $\mathrm{orb}_n(C + A, \psi) \neq \mathrm{orb}_n(C, \psi_j) + \mathrm{orb}_n(A, \psi_j)$*, if* $n \geq 2$*.*

Proof. a) Let $(\varphi_0, \varphi_1, \dots, \varphi_n) \in D(N_n)$, then

$$N_n(\varphi_0, \varphi_1, \dots, \varphi_n) = (N\varphi_0, N\varphi_1, \dots, N\varphi_n)$$
$$= (CA\varphi_0, CA\varphi_1, \dots, CA\varphi_n) = C_n(A\varphi_0, A\varphi_1, \dots, A\varphi_n)$$
$$= C_n A_n(\varphi_0, \varphi_1, \dots, \varphi_n).$$

b) Taking into account that $N\psi_j = CA\psi_j = C(\sqrt{j}\psi_{j-1}) = j\psi_j$, $j \in \mathbb{N}_0$, we have

$$\begin{aligned}
&N_n \mathrm{orb}_n(A, \psi_j)\\
&= C_n A_n(\psi_j, \sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \dots, \sqrt{j}\sqrt{j-1}\dots, \sqrt{j-n+1}\psi_{j-n})\\
&= (N\psi_j, N\sqrt{j}\psi_{j-1}, N\sqrt{j}\sqrt{j-1}\psi_{j-2}, \dots,\\
&\quad N\sqrt{j}\sqrt{j-1}\dots, \sqrt{j-n+1}\psi_{j-n})\\
&= (j\psi_j, (j-1)\sqrt{j}\psi_{j-1}, (j-2)\sqrt{j}\sqrt{j-1}\psi_{j-2}, \dots,\\
&\quad (j-n)\sqrt{j}\sqrt{j-1}\dots\sqrt{j-n+1}\psi_{j-n}),\\
&\psi_{j-n}(x) = 0, \quad \text{if } \ j < n;
\end{aligned}$$

c)

$$\begin{aligned}
&N_n \mathrm{orb}_n(C, \psi_j)\\
&= C_n A_n(\psi_j, \sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \dots,\\
&\quad \sqrt{j+1}\sqrt{j+2}\dots\sqrt{j+n}\psi_{j+n})\\
&= (N\psi_j, N\sqrt{j+1}\psi_{j+1}, N\sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \dots,\\
&\quad N\sqrt{j+1}\sqrt{j+2}\dots\sqrt{j+n}\psi_{j+n})\\
&= (j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}(j+2)\psi_{j+2}, \dots,\\
&\quad \sqrt{j+1}\sqrt{j+2}\dots\sqrt{j+n}(j+n)\psi_{j+n}).
\end{aligned}$$

d) The proof is clear. □

We prove now the generalized canonical commutation relations between operators C_n and A_n. These relations, in the case $n = 0$ coincide with the classical one.

Theorem 6.3.2. *For the commutator $[A_n, C_n] = A_nC_n - C_nA_n$ the following relations are hold:*

a) *If $(\varphi_0, \varphi_1, \dots, \varphi_n) \in D([A_n, C_n]) = D(A_nC_n) \cap D(C_nA_n)$, then*

$$A_nC_n(\varphi_0, \varphi_1, \dots, \varphi_n) - C_nA_n(\varphi_0, \varphi_1, \dots, \varphi_n) = (\varphi_0, \varphi_1, \dots, \varphi_n).$$

b) *If $\mathrm{orb}_n(A, \varphi) \in D(C_nA_n)$ and $\mathrm{orb}_n(C, \varphi) \in D(A_nC_n)$, then*

$$\begin{aligned}
&A_nC_n\mathrm{orb}_n(C, \varphi) - C_nA_n\mathrm{orb}_n(A, \varphi\\
&= (I\psi, AC^2\varphi - CA^2\varphi, \dots, AC^{n+1}\varphi - CA^{n+1}\varphi).
\end{aligned}$$

c) *If* $\mathrm{orb}_n(A,\varphi) \in D(A_nC_n)$ *and* $\mathrm{orb}_n(C,\varphi) \in D(C_nA_n)$, *then*

$$A_nC_n\mathrm{orb}_n(C,\varphi) - C_nA_n\mathrm{orb}_n(A,\varphi)$$
$$= (I\psi, ACA\varphi - CAC\varphi, \dots, ACA^n\varphi - CAC^n\varphi).$$

Proof. a) $A_nC_n(\varphi_0,\varphi_1,\dots,\varphi_n) - C_nA_n(\varphi_0,\varphi_1,\dots,\varphi_n)$

$$= A_n(C\varphi_0, C\varphi_1,\dots,C\varphi_n) - C_n(A\varphi_0, A\varphi_1,\dots,A\varphi_n)$$
$$= (AC\varphi_0, AC\varphi_1,\dots,AC\varphi_n) - (CA\varphi_0, CA\varphi_1,\dots,CA\varphi_n)$$
$$= ((AC-CA)\varphi_0, (AC-CA)\varphi_1,\dots,(AC-CA)\varphi_n) = (\varphi_0,\varphi_1,\dots,\varphi_n).$$

b) $A_nC_n\mathrm{orb}_n(C,\varphi) - C_nA\mathrm{orb}_n(A,\varphi)$

$$= A_nC_n(\varphi, C\varphi, C^2\varphi,\dots,C^n\varphi) - C_nA_n(\varphi, A\varphi, A^2\varphi,\dots,A^n\varphi)$$
$$= A_n(C\varphi, C^2\varphi,\dots,C^{n+1}\varphi) - C_n(A\varphi, A^2\varphi,\dots,A^{n+1}\varphi)$$
$$= (AC\varphi - CA\varphi, AC^2\varphi - CA^2\varphi,\dots,AC^{n+1}\varphi - CA^{n+1}\varphi)$$
$$= (I\varphi, AC^2\varphi - CA^2\varphi,\dots,AC^{n+1}\varphi - CA^{n+1}\varphi).$$

Analogously will be proved statement

c) $A_nC_n\mathrm{orb}_n(A,\varphi) - C_nA_n\mathrm{orb}_n(C,\varphi)$

$$= A_nC_n(\varphi, A\varphi, A^2\varphi,\dots,A^n\varphi) - C_nA_n(\varphi, C\varphi, C^2\varphi,\dots,C^n\varphi)$$
$$= A_n(C\varphi, CA\varphi,\dots,CA^n\varphi) - C_n(A\varphi, AC\varphi,\dots,AC^n\varphi)$$
$$= (AC\varphi - CA\varphi, ACA\varphi - CAC\varphi,\dots,ACA^n\varphi - CAC^n\varphi)$$
$$= (I\varphi, ACA\varphi - CAC\varphi,\dots,ACA^n\varphi - CAC^n\varphi).$$

The statements a) and b) give the direct generalizations of canonical commutation relation. The statements c) and d) also are generalizations of canonical commutation relation. □

Corollary. *From statement a) of Theorem 6.3.2 it follows that:*

a) *If* $\mathrm{orb_n}(C,\varphi) \in D([A_n, C_n]) = D(C_nA_n) \cap D(A_nC_n)$, *then*

$$A_nC_n\mathrm{orb_n}(C,\varphi) - C_nA_n\mathrm{orb_n}(C,\varphi) = \mathrm{orb_n}(C,\varphi).$$

b) *If* $\mathrm{orb_n}(A,\varphi) \in D(A_nC_n - C_nA_n) = D(C_nA_n) \cap D(A_nC_n)$, *then*

$$A_nC_n\mathrm{orb_n}(A,\varphi) - C_nA_n\mathrm{orb_n}(A,\varphi) = \mathrm{orb_n}(A,\varphi).$$

c) $[N_n, C_n] = C_n$ *and* $[N_n, A_n] = -A_n$.

According to well-known distributional property, we have

$$[N_n, C_n] = [C_n A_n, C_n] = C_n[A_n, C_n] + [C_n, C_n]A_n = C_n.$$

As well

$$[N_n, A_n] = [C_n A_n, A_n] = C_n[A_n, A_n] + [C_n, A_n]A_n = -A_n.$$

If we introduce in $D(C^n)$ the inner product

$$\begin{aligned}&\langle \mathrm{orb}_n(C,\varphi), \mathrm{orb}_n(C,\chi)\rangle_n \\ &= (\varphi,\chi) + (C\varphi, C\chi) + \cdots + (C^n\varphi, C^n\chi), \quad n \in \mathbb{N}_0,\end{aligned} \tag{6.3.8}$$

and the corresponding norm

$$\|\mathrm{orb}_n(C,\varphi)\|_n = (\|\varphi\|^2 + \|C^2\varphi\|^2 + \cdots + \|C^n\varphi\|^2)^{1/2}, \tag{6.3.9}$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ are inner product and norm in the space $L^2(\mathbb{R})$, then it will turn into a prehilbert space. The same can be said about $D(A^n)$. The operator C_n is linear unbounded operator in the space $D(C^n)$ with a dense image. Analogously is defined the Hilbert space $D(A^n)$ in which, the inner product and the norm are defined by formulas (6.3.8) and (6.3.9) with the replacement of C by A. The spaces $D(C^n)$ and $D(A^n)$ can be turned into Hilbert spaces by changing the domains of the operators A and C. Namely, as the domain of definition of the operators (6.3.1) and (6.3.2) we consider the set $U \cap V$. The set U consits of all functions $\varphi \in L^2(\mathbb{R})$ which are absolutely s on every finite interval on $\mathbb{R}$ and such that $\varphi' \in L^2(\mathbb{R})$. The set V consists of all functions $\psi \in L^2(\mathbb{R})$ such that $x\psi(x) \in L^2(\mathbb{R})$. It is well-known that the operator $i\frac{d}{dx}$ with the domain of definition U is selfadjoint ([96], p. 396). Taking into account that a function $\varphi \in U$ satisfy the equality $\varphi(-\infty) = \varphi(\infty) = 0$ ([96], p. 394), we verify that the operators $\frac{d}{dx}$ and $-\frac{d}{dx}$ with the domain of definition U are conjugate with each other. If we take into account yet selfadjointness of the position operator of quantum mechanics $X\psi(x) = x\psi(x),\ \psi \in V$, we obtain that the annihilation and creation operators (6.3.1) and (6.3.2) with the domain of definition $U \cap V$, are conjugate with each other. Every adjoint operator is closed ([96], p. 353). Therefore, the operators A and C with the domain of definition $U \cap V$ are closed and we can turn $D(C^n)$ into a Hilbert space with the inner product (6.3.8) and corresponding norm (6.3.9). The same can be said about $D(A^n)$.

Theorem 6.3.3. *If as the domain of definition of the operators* (6.3.1) *and* (6.3.2) *is considered the set $U \cap V$, then the sequence $\{\mathrm{orb_n}(A,\psi_k)\}$ (resp. $\{\mathrm{orb_n}(C,\psi_k)\}$), $n,k \in \mathbb{N}_0$, is an orthogonal basis in $D(A^n)$, (resp. in $D(C^n)$).*

Proof. We prove theorem for the Annihilation operator A (for the Creation operator C the proof is carried out in a similar way). The orthogonality of the sequence $\{\mathrm{orb_n}(A, \psi_k)\}$ in the space $D(A^n)$ follows from the orthogonality of $\{\psi_k(x)\}$ in $L^2(\mathbb{R})$ and from the formulae (6.3.5) and (6.3.7). Because of the sequence $\{\psi_k(x)\}$ is a basis in $L^2(\mathbb{R})$, we have for $\psi(x) \in L^2(\mathbb{R})$ that

$$\psi(x) = \sum_{k=0}^{\infty} a_k \psi_k(x),$$

where

$$a_k = \int\limits_{\mathbb{R}} \psi(x)\overline{\psi_k(x)}dx,\ k \in \mathbb{N}_0.$$

Because of $A^j\psi \in L^2(\mathbb{R}),\ j = 1, 2, \ldots, n$, we have for $\psi \in L^2(\mathbb{R})$

$$A^j\psi(x) = \sum_{k=0}^{\infty} b_k^j \psi_k(x),$$

where

$$b_k^j = \int\limits_{\mathbb{R}} A^j\psi(x)\overline{\psi_k(x)}dx.$$

Due to the fact that the operators A and C are mutually conjugate to each other, we obtain

$$b_k^j = \int\limits_{\mathbb{R}} \psi(x)C^j\overline{\psi_k(x)}dx.$$

In its turn

$$C^j\psi_k = C^{j-1}C\psi_k = C^{j-1}\sqrt{k+1}\psi_{k+1} = \sqrt{k+1}\sqrt{k+2}\ldots\sqrt{k+j}\psi_{k+j}.$$

Therefore

$$\begin{aligned} b_k^j &= \sqrt{(k+1)(k+2)\ldots(k+j)} \int\limits_{\mathbb{R}} \psi(x)\overline{\psi_{k+j}(x)}dx \\ &= \sqrt{(k+1)(k+2)\ldots(k+j)}a_{k+j} \end{aligned}$$

and

$$\begin{aligned} A^j\psi(x) = \sum_{k=0}^{\infty} \sqrt{(k+1)(k+2)\ldots(k+j)}a_{k+j}\psi_k(x) &= \sum_{k=0}^{\infty} a_{k+j}A^j\psi_{k+j}(x) \\ &= \sum_{k=j}^{\infty} a_k A^j\psi_k(x). \end{aligned}$$

We have that

$$\begin{aligned} A^j\psi_k &= A^{j-1}(A\psi_k) = \sqrt{k}A^{j-1}\psi_{k-1} = \sqrt{k(k-1)}A^{j-2}\psi_{k-2} \\ &= \sqrt{k!}A^{j-k}\psi_0,\ j-k \geq 1. \end{aligned}$$

But

$$A\psi_0 = \mathrm{const}\Big(\frac{d}{dx}e^{-x^2/4} + \frac{x}{2}e^{-x^2/4}\Big) = 0.$$

Thus $A^j\psi_k(x) = 0$, if $k = 0, 1, \ldots, j-1$, and it is proved that

$$A^j\psi(x) = \sum_{k=0}^{\infty} a_k A^j\psi_k(x).$$

Therefore, for the $\mathrm{orb_n}(A, \psi)$ the following representation is valid

$$\mathrm{orb_n}(A, \psi) = \sum_{n=0}^{\infty} a_k \mathrm{orb_n}(A, \psi_k).$$

This equality proves the Theorem 6.3.3. □

6.3.2 Orbital operators corresponding to the creation and annihilation operators in the Fréchet–Hilbert space of all orbits

In this section the orbital operators corresponding to the creation and annihilation operators in the Fréchet–Hilbert space of all orbits are constructed. It is easy to prove that algebraically $D(H^\infty) \subset D(C^\infty) \subset D(C^n)$, where C is the creation operator and H is the hamiltonian of QHO. $D(H^\infty)$ is isomorphic to the Schwartz space of rapidly decreasing functions [170] and is nonempty set of second category. The topology of the space $D(C^\infty)$ is generated with the sequence of norms (6.3.9). As well $D(C^\infty)$ is also the domain of definition of the operator C^∞ defined by equality

$$\begin{aligned} C^\infty(\varphi(x), C\varphi(x), \ldots, C^{n-1}\varphi(x), \ldots) &= \mathrm{orb}(C, C\varphi) \\ &= (C\varphi(x), C^2\varphi(x), \ldots, C^{n+1}\varphi(x), \ldots), \end{aligned} \tag{6.3.10}$$

It will be also noted that the space $D(C^\infty)$ is represented as projective limit of the sequence of the Hilbert spaces $\{D(C^n)\}$ and is Fréchet–Hilbert space.

Problem 6.3.1. It is not known whether the Fréchet space $D(C^\infty)$ is nuclear and countable-Hilbert.

In the case of annihilation operator A analogously is defined the space of all orbits $D(A^\infty)$. The space $D(A^\infty)$ is also the domain of definition of the operator A^∞ defined by equality

$$A^\infty(\varphi, A\varphi, \dots, A^n\varphi, \dots) = (A\varphi, A^2\varphi, \dots, A^{n+1}\varphi, \dots), \tag{6.3.11}$$

This means that $A^\infty(\varphi, A\varphi, \dots, A^n\varphi, \dots) = (d/dx + x/2)^\infty \mathrm{orb}(A, \varphi)$, where the operator $A^\infty \mathrm{orb}(A, \varphi)$ is really defined by equality

$$(d/dx + x/2)^\infty \mathrm{orb}(A, \varphi) \\ = ((d/dx + x/2)\varphi, (d/dx + x/2)^2\varphi, \dots, (d/dx + x/2)^{n+1}\varphi, \dots).$$

According to the statement a) of Theorem 6.3.1, we have

$$N^\infty \mathrm{orb}(C, \psi_j) = C^\infty A^\infty \mathrm{orb}(C, \psi) = (\frac{x^2}{4} - \frac{d^2}{dx^2} + \frac{1}{2})^\infty \mathrm{orb}(C, \psi_j) \\ = (j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1}, \dots, \sqrt{j+1}\sqrt{j+2}\dots\sqrt{j+n}(j+n)\psi_{j+n}, \dots).$$

Problem 6.3.2. It is not known whether the LCS $D(A^\infty)$ is nuclear and countable-Hilbert.

Theorem 6.3.4. *For the commutator* $[A^\infty, C^\infty] = A^\infty C^\infty - C^\infty A^\infty$, *where* C^∞ *and* A^∞ *are defined, respectivaly, by equalities* (6.3.10) *and* (6.3.11)*, the following relations are hold:*

a) *If* $(\varphi_0, \dots, \varphi_n, \dots) \in D([A^\infty, C^\infty])$*, then*

$$[A^\infty, C^\infty](\varphi_0, \dots, \varphi_n, \dots) = (\varphi_0, \dots, \varphi_n, \dots).$$

b) *If* $\mathrm{orb}(A, \varphi) \in D(C^\infty A^\infty)$ *and* $\mathrm{orb}(C, \psi) \in D(A^\infty C^\infty)$*, then*

$$(A^\infty C^\infty \mathrm{orb}(C, \varphi) - C^\infty A^\infty \mathrm{orb}(A, \varphi) \\ = (I\varphi, AC^2\varphi - CA^2\varphi, \dots, AC^{n+1}\varphi - CA^{n+1}\varphi, \dots).$$

c) *If* $\mathrm{orb}(A, \varphi) \in D(A^\infty, C^\infty)$ *and* $\mathrm{orb}(C, \varphi) \in D(C^\infty A^\infty)$*, then*

$$A^\infty C^\infty \mathrm{orb}(A, \varphi) - C^\infty A^\infty \mathrm{orb}(C, \varphi) \\ = (I\varphi, ACA\varphi - CAC\varphi, \dots, ACA^n\varphi - CAC^n\varphi, \dots).$$

The statement a) gives us the direct generalization of canonical commutation relation. The statements b) and c) also are generalization of canonical commutation relation.

Corollary. *From the statement* a) *of Theorem* 6.3.4 *it follows that*

a) *If* $\operatorname{orb}(A,\varphi) \in D([A^\infty, C^\infty])$, *then*

$$[A^\infty, C^\infty]\operatorname{orb}(A,\varphi) = \operatorname{orb}(A,\varphi).$$

b) *If* $\operatorname{orb}(C,\varphi) \in D([A^\infty, C^\infty])$, *then*

$$[A^\infty, C^\infty]\operatorname{orb}(C,\varphi) = \operatorname{orb}(C,\varphi).$$

c) $[N^\infty, C^\infty] = C^\infty$ *and* $[N^\infty, A^\infty] = A^\infty$.

Really, according distributivity property, we have

$$[N^\infty, C^\infty] = [C^\infty A^\infty, C^\infty] = C^\infty[A^\infty, C^\infty] + [C^\infty, C^\infty]A^\infty = C^\infty.$$

As well

$$[N^\infty, A^\infty] = [C^\infty A^\infty, A^\infty] = C^\infty[A^\infty, A^\infty] + [C^\infty, A^\infty]A^\infty = -A^\infty.$$

6.4 Central spline algorithms for calculation of the inverse of one dimensional hamiltonian of QHO on Schwartz space

In this section H denotes a separable complex Hilbert space with the norm $\|\cdot\|$ generated by inner product $(\cdot,\cdot)$. Let $A : D(A) \subset H \to H$ be a linear, symmetric, positive operator with a discrete spectrum of positive eigenvalues and dense image. The spectrum of A is called discrete, if it consists of a countable set of eigenvalues with a single limit point at infinity.

Let $n \in \mathbb{N}_0$ be a fixed nonnegative whole number and consider the elements of the space H, to which the operator $A^n = A(A^{n-1})$ can be applied, where A^0 is the identical operator. The space of such elements is denoted by $D(A^n)$, besides, $D(A^0) = H$. By n-orbits of the operator A at the point $x \in H$ we mean a finite sequence $\operatorname{orb}_n(A,x) := (x, Ax, \ldots, A^n x)$, $n \in \mathbb{N}_0$. The space $D(A^n)$ we identify with the space of n-orbits of the operator A. We can turn $D(A^n)$ into a pre-Hilbert space using the inner product

$$\begin{aligned}\langle \operatorname{orb}_n(A,x), \operatorname{orb}_n(A,y)\rangle_n &:= (x,y)_n \\ &= (x,y) + (Ax,Ay) + \cdots + (A^n x, A^n y), \quad n \in \mathbb{N}_0. \qquad (6.4.1)\end{aligned}$$

We consider the equation $Au = f$ in the space $D(A^n)$, in fact, it has the form

$$A_n(\operatorname{orb}_n(A,u)) = \operatorname{orb}_n(A,f), \qquad (6.4.2)$$

where the operator $A_n : D(A_n) = (D(A))^{n+1} \subset H^{n+1} \to \operatorname{Im} A_n = (\operatorname{Im} A)^{n+1} \subset H^{n+1}$ is defined by the equality

$$A_0(u) = u, \quad A_n(\operatorname{orb}_n(A,u)) = \operatorname{orb}_n(A, Au), \quad n \geq 1. \tag{6.4.3}$$

We call A_n as n-orbital operator, which corresponds to the operator A, and (6.4.2) is called the n-orbital equation in the space $D(A^n)$.

For the approximation solution of the equation (6.4.2), a linear central spline algorithm in the space $D(A^n)$ is constructed in Section 1.4 (Theorem 1.4.2). The convergence of the sequence of approximate solutions to the generalized solution is proved.

It was arisen the problem of construction of linear central spline algorithms in Fréchet spaces of all orbits. The definitions of linear central spline algorithms, which is given in ([158], Sections 5.7 and 3.2), are generalized in Chapter 3 of this monography for solution operators acting in locally convex metrizable spaces. For the case of a linear normed space, our definitions of spline and central algorithms coincide with those given in [158]. For nonnormable spaces the study of such questions was begun in ([157], Chapter 1).

We transfer the equation $Au = f$ also in the Fréchet space of all orbits $D(A^\infty)$, which is represented as projective limit of the sequence of the spaces $\{D(A^n)\}$. Really, the space $D(A^\infty) = \cap_{n=0}^{\infty} D(A^n)$, and the topology of these spaces are generated with the sequence of norms

$$\|x\|_n = \left(\|x\|^2 + \|Ax\|^2 + \cdots + \|A^n x\|^2\right)^{1/2}, \quad n \in \mathbb{N}_0. \tag{6.4.4}$$

The space $D(A^\infty)$ is isomorphic to the space of all orbits $\operatorname{orb}(A,x)$ of operator A. This isomorphism (actually it is an isometry) is obtained by the mapping

$$D(A^\infty) \ni x \to \operatorname{orb}(A,x) := \left(x, Ax, \ldots, A^n x, \ldots\right).$$

The space $D(A^\infty)$ circumstantially was studied in ([160], Chapter 8), where $D(A^\infty)$ was whole symbol and A^∞ taken separately meant nothing. In [163], we have defined the operator A^∞ as follows:

$$A^\infty\left(x, Ax, \ldots, A^{n-1}x, \ldots\right) = \left(Ax, A^2 x, \ldots, A^n x, \ldots\right). \tag{6.4.5}$$

Due to this notation, now $D(A^\infty)$ is the domain of definition of the operator A^∞. It will be also noted that, according to statement d) of Theorem 4.4.2, the operator A^∞ is selfadjoint and positive definite in the Fréchet space $D(A^\infty)$. The operator A^∞ is defined on the whole space and, by generalized Hellinger–Teoplitz theorem, is continuous. Moreover, A^∞ is a topological isomorphism onto the space $D(A^\infty)$, and the equation

$$A^\infty \operatorname{orb}(A,u) = \operatorname{orb}(A,f) \tag{6.4.6}$$

admits a unique stable solution $(A^\infty)^{-1}\operatorname{orb}(A,f) = S(\operatorname{orb}(A,f))$, where $S = (A^\infty)^{-1}$ is an inverse operator to A^∞.

In Section 6.4.2, for the approximation solution of the equation (6.4.6), a linear central spline algorithm in the space $D(A^\infty)$ is constructed. The convergence of the sequence of approximate solutions to the exact solution is proved. In Section 6.4.3, we apply the results obtained for the general operators to the quantum harmonic oscillator operator in the spaces of orbits.

6.4.1 Construction of a linear central spline algorithms in the Fréchet space of all orbits

Let F_1 be a complex linear space with a nonincreasing sequence of absolutely convex absorbed sets $\{V_n\}$. We denote by F_1 the metrizable LCS which is generated by a nondecreasing sequence of seminorms $\{\|\cdot\|_n\}$ and $V_n = \{f \in E,\ \|f\|_n \le 1\}$, i.e. $\|\cdot\|_n$ is the Minkowski functional for V_n.

D. Zarnadze constructed a metric with absolutely convex balls having the following form:

$$d(x,y) = \begin{cases} \|x-y\|_1 & \text{if } \|x-y\|_1 \ge 1, \\ 2^{-n+1} & \text{if } \|x-y\|_n < 2^{-n+1} \text{ and} \\ & \qquad \|x-y\|_{n+1} \ge 2^{-n+1}\ (n \in \mathbb{N}), \\ \|x-y\|_{n+1} & \text{if } 2^{-n} \le \|x-y\|_{n+1} < 2^{-n+1}\ (n \in \mathbb{N}), \\ 0 & \text{if } x-y=0. \end{cases} \tag{6.4.7}$$

The Minkowski functionals $q_r(\cdot)$ for the balls K_r of the metric (6.4.7) are dependent on the initial seminorms through the following simple equality:

$$q_r(\cdot) = r^{-1}\|\cdot\|_n, \quad \text{where } \ r \in I_n = \begin{cases} [1,\infty[& \text{if } n=1, \\ [2^{-n+1}, 2^{-n+2}[& \text{if } n \ge 2. \end{cases} \tag{6.4.8}$$

Thus $K_r = rV_n$, where $V_n = \{x \in E; \|x\|_n \le 1\}$, and $r \in I_n$. From this it also follows that $\|x\|_n = |x|$, when $|x| \in \operatorname{Int} I_n$, and $\|x\|_n \le |x|$, when $|x| = 2^{-n+1}$, $n \in \mathbb{N}$, where $|x| = d(x,0)$ is quasinorm of metric d. Similarly to the case of a normed linear space, K_r are simply expressed with the unit balls of the topology generating seminorms (6.4.8). Defined by the metric (6.4.7), the balls K_r preserve the geometry of the initial space.

The definitions of spline and spline algorithm for a solution operator $S : F_1 \to E$, where E is metrizable LCS with the non-increasing sequence of absolute convex absorbed sets $V_n = \{x \in E; \|x\|_n \le 1\}$, $n \in \mathbb{N}$, and metric (6.4.7). Consider the set $F = \{f \in F_1 :\ d(f,0) \le 1\}$.

Let $\varepsilon \geq 0$. It is known that $U(f) = \varphi(If)$ is an ε-approximation of $S(f)$ iff $|S(f) - U(f)| \leq \varepsilon$. Let

$$\begin{aligned} r_1(I, y) &= \mathrm{rad}(S(I^{-1}(y) \cap F) \\ &= \inf\{\sup\{|S(f) - g|;\ \ f \in I^{-1}(y) \cap F\};\ g \in E\} = \varepsilon \in \mathrm{int} I_n. \end{aligned}$$

According to the definition of Chebyshev radius, this means that K_ε is smallest ball that contains the set $S(I^{-1}(y) \cap F)$. But $K_\varepsilon = \varepsilon V_n$ according (6.4.8) and this coincides with the ε ball of norm $\|\cdot\|_n$ when $\varepsilon \in \mathrm{int} I_n$. This means also that

$$\begin{aligned} &\mathrm{rad}(S(I^{-1}(y) \cap F) \\ &\qquad = \inf\{\sup\{\|S(f) - g\|_n;\ f \in I^{-1}(y) \cap F\};\ \ g \in E\} = \varepsilon \in \mathrm{int} I_n. \end{aligned}$$

We obtain that the ε-approximation of $S(f)$ with respect to the metric d and with respect to norm $\|\cdot\|_n$ is the same. As is known, ε-complexity is the minimal cost of computing of ε-approximation with a prescribed ε-accuracy. Therefore, the ε-complexity of problem with respect to the metric can be calculated by means of ε-complexity with respect to norm.

Let now $T : D(T) \subset E \to E$ be a linear operator, where E is a metric lcs. An operator $S : E \to E$ is said to be the solution operator of an operator equation $Tu = f$ if $u = Sf$. If there exists an inverse to T, then $S = T^{-1}$. In addition, the central (resp., linear, spline, optimal) algorithm, approximating the solution operator S, will be called the central (resp., linear, spline, optimal) algorithm for the equation $Tu = f$.

It will be noted that in each nonnormable Fréchet space there exists an one-dimensional subspace that admits no orthogonal complement subspace. In particular, this is valid for the space $D(A^\infty)$ with the sequence of norms (6.4.4). This means that in the Fréchet space $D(A^\infty)$ there exists one-dimensional subspace that admits no orthogonal complement subspace.

In the case $E = D(A^\infty)$, the operator K_n, considered in Theorem 3.6.2, is the identical map from $D(A^\infty)$ in $(D(A^n), \|\cdot\|_n)$, i.e. K_n is defined by equality $K_n(\mathrm{orb}(A, x)) = \mathrm{orb}_n(A, x)$. Let π_{mn} be an identical map $\pi_{mn} : (D(A^m), \|\cdot\|_m) \to (D(A^n), \|\cdot\|_n)$, i.e. π_{mn} is defined by equality $\pi_{mn}(\mathrm{orb}_m(A, x)) = \mathrm{orb}_n(A, x)$ $(n \leq m)$. The space $D(A^\infty)$ is a projective limit of the sequence of prehilbert spaces $\{(D(A^n), \|\cdot\|_n)\}$ with respect to the maps π_{mn} $(n \leq m)$. This means that the topology of Fréchet space is the weakest topology for which the maps K_n are continuous ([82], p. 232). Also, each element $\mathrm{orb}(A, f) \in D(A^\infty)$ can be identified with the sequence $\{K_n \,\mathrm{orb}(A, f)\} = \{\mathrm{orb}_n(A, f)\}$. In particular, the element $\mathrm{orb}(A, f)$ can be identified with the sequence $\{K_n \,\mathrm{orb}(A, f)\} =$

$\{\mathrm{orb}_n(A,f)\}$. It is simply to verify that the sequence $\{\mathrm{orb}_n(A,f)\}$ converges to the $\mathrm{orb}(A,f)$.

Projective operator A_n^∞ of the operator A^∞ on the Hilbert space $(D(A^n), \|\cdot\|_n)$ is defined by the equality

$$A_n^\infty(K_n \,\mathrm{orb}(A,x)) = K_n(A^\infty \,\mathrm{orb}(A,x)) = (Ax, A^2x, \dots, A^{n+1}x).$$

It is not difficult to verify that the operator A_n^∞ coincides with the operator A_n defined by equality (6.4.3).

6.4.2 Construction of a linear central spline algorithm for equations containing QHO operator in the Fréchet–Hilbert spaces of all orbits

The quantum harmonic oscillator operator $Au(t) = -u''(t) + t^2u(t)$, $t \in \mathbb{R}$, in the Hilbert space of finite orbits $D(A^n)$ we have considered in Section 1.4.

Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ with the usually used sequence of norms

$$\|\varphi\|_{n,m} = \sup_{t\in\mathbb{R}}\{\|t^m d^n/dt^n \varphi(t)\|,\ n,m \in \mathbb{N}\}.$$

According to ([138], Addition to Section 5.3), the spaces $\mathcal{S}(\mathbb{R})$ and $D(A^\infty)$ coincide and the sequences of norms $\{\|\cdot\|_n\}$ and $\{\|\cdot\|_{n,m}\}$ generate equivalent topologies.

The operator A^∞ is the restriction of the considered by us QHO operator A on the space $D(A^\infty)$, taking into account the topology (really, the operator A^∞ is the restriction of the operator $A^\mathbb{N}$ from the Fréchet space $(L^2(\mathbb{R}))^\mathbb{N}$ on the Fréchet space $D(A^\infty)$). According to the equality $D(A^\infty) = \mathcal{S}(\mathbb{R})$, the operator A^∞ is also a restriction on the space $\mathcal{S}(\mathbb{R})$, taking into account the topology. The QHO operator on the space $\mathcal{S}(\mathbb{R})$ was also considered in [13].

It follows from the statement d) of Theorem 4.4.3 that the equation (6.4.6) has the unique and stable solution. To solve the equation (6.4.6) approximately in the space $D(A^\infty) = \mathcal{S}(\mathbb{R})$, we construct a spline for the nonadaptive information $I(f) = [L_1(f), \dots, L_m(f)]$, where $L_i(f) = (\mathrm{orb}(A,f), \mathrm{orb}(A,h_i))$, $i = 0,1,\dots,m$, are continuous functionals on $\mathcal{S}(\mathbb{R})$, where $\{h_k\}$, $k \in \mathbb{N}$, is the orthogonal basis on $L^2(\mathbb{R})$ defined by (1.4.10). The subspace $\mathrm{Ker}\, I$ is a finite codimensional subspace in the space $D(A^\infty)$. The spline $\mathrm{orb}(A,\sigma_m)$ interpolatory $y = I(f)$ is given by the equality (1.4.6). It is the best approximation element of $\mathrm{orb}(A,f)$ in the subspace

$$\mathrm{span}\{\mathrm{orb}(A,h_0), \mathrm{orb}(A,h_1), \dots, \mathrm{orb}(A,h_m)\}$$

with respect to the norm (6.4.4) and coefficients in (1.4.6) do not depend on n. This means that the subspace $\operatorname{Ker} I$ admits an orthogonal complement subspace

$$\operatorname{Ker} I^{\perp} = \operatorname{span}\{\operatorname{orb}(A, h_0), \operatorname{orb}(A, h_1), \ldots, \operatorname{orb}(A, h_m)\}$$

in the Fréchet space $D(A^{\infty})$. Consider the spline algorithm

$$\operatorname{orb}(A, u_m) = S(\operatorname{orb}(A, \sigma_m)) = \sum_{k=0}^{m} \lambda_k^{-1}(f, h_k) \operatorname{orb}(A, h_k), \tag{6.4.9}$$

where S is the solution operator of the equation (6.4.6), i.e. the inverse of the operator A^{∞} in the space $D(A^{\infty})$.

Theorem 6.4.1. *Let $A : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the QHO operator acting in the Hilbert space $L^2(\mathbb{R})$ and A^{∞} be the operator* (6.4.5) *acting in the space of all orbits $D(A^{\infty}) = \mathcal{S}(\mathbb{R})$. Then the algorithm* (6.4.9) *is a linear central spline algorithm for the approximate solution of the orbital equation* (6.4.6) *in the space $\mathcal{S}(\mathbb{R})$. The sequence $\{\operatorname{orb}(A, u_m)\}$ of the approximate solutions* (6.4.9) *converges to the exact solution $\operatorname{orb}(A, u_0)$ of the orbital equation* (6.4.6) *in the space $D(A^{\infty}) = \mathcal{S}(\mathbb{R})$. Moreover, the following estimates hold:*

a) *For every n and m,*

$$\| \operatorname{orb}(A, u_0) - \operatorname{orb}(A, u_m)\|_n \leq \| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_n;$$

b) *For every $n \in \mathbb{N}_0$ there exists $m_0 = m_0(n)$ such that for every $m > m_0(n)$,*

$$\| \operatorname{orb}(A, u_0) - \operatorname{orb}(A, u_m)\|_n \leq | \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)|;$$

c) *For every m,*

$$| \operatorname{orb}(A, u_0) - \operatorname{orb}(A, u_m)| \leq | \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)|.$$

Proof. According to the statement b) of Theorem 3.6.2, $\operatorname{orb}(A, \sigma_m)$ is a center or all sets $I^{-1}(y) \cap V_k$, for which this intersections are non-empty. The correspondence spline algorithm $\varphi^s(y) = S(\operatorname{orb}(A, \sigma_m))$ is linear and central in the Fréchet space $D(A^{\infty})$. It is simply to matter that the sequence of approximate solutions $\{(\operatorname{orb}(A, u_m)\}$ converges to $\sum_{k=0}^{\infty} \lambda_k^{-1}(f, h_k) \operatorname{orb}(A, h_k)$. From the existence and uniqueness of the exact solution it follows that

$$\operatorname{orb}(A, u_0) = S(\operatorname{orb}(A, f)) = \sum_{k=0}^{\infty} \lambda_k^{-1}(f, h_k) \operatorname{orb}(A, h_k),$$

i.e. the sequence of the approximate solutions $\{\operatorname{orb}(A, u_m)\}$ converges to the exact solution in the space $D(A^\infty)$.

a) We have that if $f \in D(A^\infty)$, then

$$\operatorname{orb}(A, f) = \sum_{k=0}^{\infty} (f, h_k) \operatorname{orb}(A, h_k)$$

and

$$\begin{aligned}\operatorname{orb}(A, Af) &= \sum_{k=0}^{\infty} (Af, h_k) \operatorname{orb}(A, h_k) = \sum_{k=0}^{\infty} (f, Ah_k) \operatorname{orb}(A, h_k)\\ &= \sum_{k=0}^{\infty} \lambda_k (f, h_k) \operatorname{orb}(A, h_k),\end{aligned}$$

where $\lambda_k = 2k+1$ is the k-th eigenvalue of QHO operator A. Then

$$\begin{aligned}&(A^\infty \operatorname{orb}(A, f), \operatorname{orb}(A, f))\\ &\quad= \Big(\sum_{k=0}^{\infty} \lambda_k (f, h_k) \operatorname{orb}(A, h_k), \sum_{k=0}^{\infty} (f, h_k) \operatorname{orb}(A, h_k) \Big)\\ &\quad\geq \lambda_0 \Big(\sum_{k=0}^{\infty} (f, h_k) \operatorname{orb}(A, h_k), \sum_{k=0}^{\infty} (f, h_k) \operatorname{orb}(A, h_k) \Big)\\ &\quad= (\operatorname{orb}(A, f), \operatorname{orb}(A, f)).\end{aligned}$$

Hence, it follows that for all $n \in \mathbb{N}_0$,

$$(A^\infty \operatorname{orb}(A, f), \operatorname{orb}(A, f))_n \geq (\operatorname{orb}(A, f), \operatorname{orb}(A, f))_n. \tag{6.4.10}$$

While

$$(A^\infty \operatorname{orb}(A, f), \operatorname{orb}(A, f))_n \leq \|A^\infty \operatorname{orb}(A, f)\|_n \| \operatorname{orb}(A, f)\|_n, \quad n \in \mathbb{N}_0,$$

from (6.4.10) it follows that $\| \operatorname{orb}(A, f)\|_n \leq \|A^\infty \operatorname{orb}(A, f)\|_n$, i.e.

$$\begin{aligned}&\| \operatorname{orb}(A, u_0) - \operatorname{orb}(A, u_m)\|_n \leq \|A^\infty \operatorname{orb}(A, u_0) - A^\infty \operatorname{orb}(A, u_m)\|_n\\ &= \|A^\infty S \operatorname{orb}(A, f) - A^\infty (\operatorname{orb}(A, u_m)\|_n = \| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_n,\end{aligned}$$

and the statement a) of Theorem 6.4.1 is proved.

b) Let $| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)| := E_m(f)$. It follows from the relation $\lim_{m\to\infty} \| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_n = 0$, $n \in \mathbb{N}_0$, that $\lim_{m\to\infty} E_m(f) = 0$. Therefore, there exists a number $m_0 = m_0(n)$ such that for every $m \geq m_0(n)$,

$E_m(f) < 2^{-n+1}$. Then $E_m(f) \in I_{p+1} = [2^{-p}, 2^{-p+1}[$ for some $n \le p$. According to (6.4.8), if $E_m(f) \in \operatorname{Int} I_{p+1}$, we obtain $\| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_p = E_m(f)$, and if $E_m(f) = 2^{-p}$, then $\| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_p \le 2^{-p}$. That is, $\| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_p \le E_m(f)$, and since $n \le p$, $\| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_n \le \| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_p = E_m(f)$ for all $m \ge m_0(n)$.

c) According to the statement a), for every n and m,

$$\| \operatorname{orb}(A, u_0) - \operatorname{orb}(A, u_m)\|_n \le \| \operatorname{orb}(A, f) - \operatorname{orb}(A, \sigma_m)\|_n.$$

From (6.4.7) and (6.4.8) follows the following property of quasinorm $|\cdot|$: if for arbitrary $u, v \in \mathcal{S}(\mathbb{R})$ the inequalities $\|u\|_n \le \|v\|_n$ hold for all $n \in \mathbb{N}$, then $|u| \le |v|$. From this property follows the statement c) of Theorem 6.4.1. □

Remark 6.4.1. The construction of central algorithm for the equation containing the Schrödinger operator in space of finite orbits is performed in Section 1.4 for the orbital equation $\mathcal{H}^\infty \operatorname{orb}(\mathcal{H}, \psi) = \operatorname{orb}(\mathcal{H}, f)$ with the orbital operator $\mathcal{H}^\infty$ for the hamiltonian in the Fréchet space of all orbits $D(\mathcal{H}^\infty)$.

6.5 Central spline algorithms for calculation of the inverse of multidimensional hamiltonian of QHO on Schwartz space

In this section the hamiltonian of the multidimensional QHO

$$\mathcal{H}_p u(t) = -\Delta_p u(t) + |t|^2 u(t) \tag{6.5.1}$$

in Hilbert space $L^2(\mathbb{R}^p)$ is considered, where $\Delta_p u(t) = \sum_{j=1}^{p} \frac{\partial^2 u}{\partial t_j^2}$ is the Laplace operator, $t = (t_1, \dots, t_p)$ and $|t|^2 = t_1^2 + t_2^2 + \cdots + t_p^2$. We shall denote p-dimensional vectors of a Euclidean space $\mathbb{R}^p$ by x, y, t, and p-dimensional multi-indices by $\mathbf{k}, \mathbf{n}$. p-dimensional multi-indices are p-dimensional vectors with whole non-negative coordinates. The set of such vectors we denoted by $\mathbf{Q}^p$. For vectors $x, y \in \mathbb{R}^p$, $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_p)$, the notions $x \le y$ (or $x \ge y$) means that $x_j \le y_j$ (or $x_j \ge y_j$) for every $j = 1, \dots, p$. Similar convention is followed for multi-indices.

The domain $D(\mathcal{H}_p)$ of the operator $\mathcal{H}_p$ is defined as the set of functions $u(t)$, $t \in \mathbb{R}^p$, that satisfy the following conditions: $u(t)$ and partial derivatives of the first order $\frac{\partial}{\partial t_j} u(t)$, $j = 1, \dots, p$, are continuous on $\mathbb{R}^p$, belong to $L^2(\mathbb{R}^p)$ and $\lim\limits_{|t|\to\infty} u(t) = 0$, $\lim\limits_{|t|\to\infty} \frac{\partial}{\partial \alpha_j} u(t) = 0$, $j = 1, \dots, p$; all partial derivatives of the second order $\frac{\partial^2}{\partial t_j^2} u(t)$, $j = 1, \dots, p$, and $|t|^2 u(t)$ belong to $L^2(\mathbb{R}^p)$. It is proved in ([137], Section 5.5, 5.13) that this condition is satisfied if: for $n = 2$ all partial derivatives of order 2 belong to $L^2(\mathbb{R}^2)$, and for $p \geq 3$ all partial derivatives of order $l \geq p/2$ belong to $L^2(\mathbb{R}^p)$.

Let us prove that the operator $\mathcal{H}_p$ is symmetric and positive on $D(\mathcal{H}_p)$. Using integrating by parts twice, for $u, v \in D(\mathcal{H}_p)$ we obtain that

$$\begin{aligned}
(\mathcal{H}_p u, v) &= \int (-\Delta_p u(t) + |t|^2 u(t), \overline{v(t)})dt \\
&= \sum_{j=1}^{p} \int -\frac{\partial^2 u(t)}{\partial t_j^2}\overline{v(t)}dt + \int |t|^2 u(t)\overline{v(t)}dt \\
&= \sum_{j=1}^{p} \int -u(t)\overline{\frac{\partial^2 v(t)}{\partial t_j^2}}dt + \int |t|^2 u(t)\overline{v(t)}dt \\
&= (u(t), \mathcal{H}_p v),
\end{aligned}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^p)$ and integral is taken on $\mathbb{R}^p$.This means that $\mathcal{H}_p$ is symmetric. Next, again using integrating by parts, we obtain

$$\begin{aligned}
(\mathcal{H}_p u, u) &= -\sum_{j=1}^{p} \int (-\Delta_p u(t) + |t|^2 u(t))\overline{u(t)}dt \\
&= \sum_{j=1}^{p} \int -\frac{\partial^2 u(t)}{\partial t_j^2}\overline{u(t)}dt + \int |t|^2 u(t)\overline{u(t)}dt \\
&= \sum_{j=1}^{p} \int -\frac{\partial u(t)}{\partial t_j}\overline{\frac{\partial u(t)}{\partial t_j}}dt + \int |t|^2 u(t)\overline{u(t)}dt \\
&= \sum_{j=1}^{p} \int \Big| \frac{\partial u(t)}{\partial t_j} \Big|^2 dt + \int |t|^2 |u(t)|^2 dt \geq 0.
\end{aligned}$$

If $(\mathcal{H}_p u, u) = 0$, then it follows from here that $u(t) \equiv 0$. This means that $\mathcal{H}_p$ is positive operator on $D(\mathcal{H}_p)$.

Let $\{h_j(t)\}$ be the sequence of the Hermitian functions, defined by (1.4.10). These functions form a complete orthogonal system in $L_2(\mathbb{R})$. For arbitrary natural number j, the function $h_j(t)$ is eigenfunction of the one-dimensional QHO $\mathcal{H}_1$ with the eigennumber $2j+1$. As $p \geq 2$ dimensional analogue of these functions, the functions $H_{\mathbf{n}}(t) = \prod_{j=1}^{p} h_{n_j}(t_j)$ are considered, where h_{n_j} is the Hermite function, $\mathbf{n} = (n_1, \ldots, n_p)$, $t = (t_1, \ldots, t_p)$, $n_j \in \mathbb{N}_0$, $t_j \in \mathbb{R}$, $j = 1, \ldots, p$. $\{H_{\mathbf{n}}(t)\}$ form a complete orthonormal system in $L^2(\mathbb{R}^p)$ and are eigenfunctions for QHO corresponding to the eigenvalue $2|\mathbf{n}| + p$, $|\mathbf{n}| = \sum_{j=1}^{p} n_j$.

Really,

$$\mathcal{H}_p H_{\mathbf{n}}(t) = \sum_{j=1}^{p} \left(\frac{\partial^2}{\partial t_j^2} + t_j^2 \right) h_{n_j}(t_j) \prod_{i=1, i\neq j}^{p} h_{n_i}(t_i)$$
$$= \sum_{j=1}^{p} (2n_j + 1) H_{\mathbf{n}}(t) = (2|\mathbf{n}| + p) H_{\mathbf{n}}(t). \qquad (6.5.2)$$

The case $p = 1$ is considered in Section 6.4 (see also [170]).

The operator $\mathcal{H}_p$ creates finite orbits $\operatorname{orb}_n(\mathcal{H}_p, \psi) = (\psi, \mathcal{H}_p\psi, \dots, \mathcal{H}_p^n\psi)$ at the function $\psi \in L^2(\mathbb{R}^p)$ and the pre-Hilbert space of finite orbits $D(\mathcal{H}_p^n)$, the topology of which is generated with the norm

$$\begin{aligned}\|u\|_n &= \| \operatorname{orb}_n(\mathcal{H}_p, u)\|_n \\ &= (\|u\|^2 + \|\mathcal{H}_p u\|^2 + \|\mathcal{H}_p^2 u\|^2 + \cdots + \|\mathcal{H}_p^n u\|^2)^{1/2} \\ &= (\|u\|^2 + \|(-\Delta_p u(t) + |t|^2 u(t))\|^2 + \cdots \\ &\quad + \|\mathcal{H}_p^{n-1}(-\Delta_p u(t) + |t|^2 u(t))\|^2)^{1/2}, \quad n \in \mathbb{N}_0, \end{aligned} \qquad (6.5.3)$$

where $\|f\| = (\int_{\mathbb{R}^p} f^2(t)dt)^{1/2}$ is a norm in the space $L_2(\mathbb{R})$. The norm $\|\cdot\|_n$ is Hilbertian and is generated by the inner product

$$\langle f, g\rangle_n = (f, g) + (\mathcal{H}_p f, \mathcal{H}_p g) + \cdots + (\mathcal{H}_p^n f, \mathcal{H}_p^n g), \qquad (6.5.4)$$

where $(f, g) = \int_{\mathbb{R}^p} f(t)g(t)dt$.

For the domain of definition $D(\mathcal{H}_p^k)$ of the operator $\mathcal{H}_p^k$, $k \in \mathbb{N}_0$, we will consider the functions u that satisfy the conditions: a) the functions u and all their partial derivatives up to the k-th order inclusive belong to $\mathbb{R}^p$; b) the functions $|t|^{2k}u(t)$ belong to $\mathbb{R}^p$; c) the functions u and all their partial derivatives up to the $(k-1)$-th order inclusive vanish at infinity. If we recall the proof of symmetry and positivity of the operator $\mathcal{H}_p$ in $D(\mathcal{H}_p)$, we will see that the operator $\mathcal{H}_p^k$ will have the same properties in the space $D(\mathcal{H}_p^k)$.

The operator $\mathcal{H}_p$ also creates the orbits $\operatorname{orb}(\mathcal{H}_p, \psi) = (\psi, \mathcal{H}_p\psi, \dots, \mathcal{H}_p^n\psi, \dots)$ at the function $\psi \in L^2(\mathbb{R}^p)$ and the Fréchet–Hilbert space $D(\mathcal{H}_p^\infty)$ of all orbits the topology of which is generated with the sequence of norms (6.5.3). The Fréchet–Hilbert space in this case coincides with the Schwartz's space $\mathcal{S}(\mathbb{R}^p)$ of rapidly decreasing functions and this isomorphism is obtained by the mapping

$$\mathcal{S}(\mathbb{R}^p) \ni x \to \operatorname{orb}_n(\mathcal{H}_p, x) := (x, \mathcal{H}_p x, \dots, \mathcal{H}_p^n x, \dots) \in D(\mathcal{H}_p^\infty).$$

The symbol $D(\mathcal{H}_p^\infty)$ means the domain of definition of the operator $\mathcal{H}_p^\infty$. It will be also noted that according to the statement d) of Theorem 4.4.3, for the positive operator $\mathcal{H}_p$, the orbital operator $\mathcal{H}_p^\infty$ is topological isomorphism of the Fréchet–Hilbert space $D(\mathcal{H}_p^\infty)$ onto itself. Also, orbital operator $\mathcal{H}_p^\infty$ coincides to the restriction of the operator $\mathcal{H}_p^{\mathbb{N}}$ from the Fréchet space $(L^2(\mathbb{R}^p))^N$ on the subspace $D(\mathcal{H}_p^\infty)$ with considered topology.

Using the above reasoning, now we define the symmetric finite orbital operators $\mathcal{H}_{pn} : D(\mathcal{H}_p^n) \to D(\mathcal{H}_p^n)$ corresponding to $\mathcal{H}_p$ by the equality

$$\mathcal{H}_{pn}(\mathrm{orb}_n(\mathcal{H}_p, u) = \mathrm{orb}_n(\mathcal{H}_p, \mathcal{H}_p u) \tag{6.5.5}$$

and the equation

$$\mathcal{H}_{pn}\,\mathrm{orb}_n(\mathcal{H}_p, u) = \mathrm{orb}_n(\mathcal{H}_p, f). \tag{6.5.6}$$

Also, we define the self-adjoint orbital operator $H_p^\infty : D(\mathcal{H}_p^\infty) \to D(\mathcal{H}_p^\infty)$ in the Fréchet–Hilbert space of all orbits, corresponding to $\mathcal{H}_p$ by equality

$$\mathcal{H}_p^\infty\,\mathrm{orb}(\mathcal{H}_p, \psi) = \mathrm{orb}(\mathcal{H}_p, \mathcal{H}_p\psi)) \tag{6.5.7}$$

and the equation

$$\mathcal{H}_p^\infty(\mathrm{orb}\,\mathcal{H}_p, u) = \mathrm{orb}(\mathcal{H}_p, f). \tag{6.5.8}$$

This means that the restriction of the equation $\mathcal{H}_p u = f$ on $\mathcal{S}(\mathbb{R}^p)$ with considering topology coincides to the equation (6.5.8). This equation, according to the statement d) of Theorem 4.4.3, has unique and stable solution. Stability is very important for numerical calculations of practical problems.

Therefore, the QHO operator considered in Hilbert space $L^2(\mathbb{R}^p)$ will be denoted by $\mathcal{H}_p$ and "apparently the same operator" considered in the Fréchet space of all orbits $D(\mathcal{H}_p^\infty)$ with considering topology will be denoted by $\mathcal{H}_p^\infty$. The equations (6.5.6) and (6.5.8) we call orbital equations.

Our goal is to built up a linear spline algorithm for the approximate solution of n-orbit equation (6.5.6) in the space $D(\mathcal{H}_p^n)$ and for equation (6.5.8) in the space $D(\mathcal{H}_p^\infty)$.

To construct an approximate solution $U(f)$ for (6.5.5), we use an information of the cardinality $\mathbf{M}$ where $\mathbf{M}$ is some subset of $\mathbf{Q}^p$. We consider the following spaces: the linear space F_1 consisting of elements of the space $D(\mathcal{H}_p^n)$; $G = D(\mathcal{H}_p^n)$. Let T be an identical operator from F_1 onto $X := (D(\mathcal{H}_p^n), \|\cdot\|_n)$. The set of problem elements is $\{f \in F_1; \|T(f)\|_n \le 1\}$. The spline interpolating the set $\{\langle f, H_{\mathbf{k}}\rangle_n,\ \mathbf{k} \in \mathbf{M}\}$ is defined as an element belonging to the space $D(\mathcal{H}_p^n)$ which is generated by an element $\sigma_{\mathbf{M}}$, $\mathbf{M} \in \mathbf{Q}^p$, satisfying the condition $\langle \sigma_{\mathbf{M}}, H_{\mathbf{k}}\rangle_n = \langle f, H_{\mathbf{k}}\rangle_n$, $\mathbf{k} \in \mathbf{M}$, and $\|T\mathrm{orb}_n(\mathcal{H}_p, \sigma_{\mathbf{M}})\|_n = \inf\{\|T(z)\|_n$, $\langle z, H_{\mathbf{k}}\rangle_n = \langle f, H_{\mathbf{M}}\rangle_n$, $\mathbf{k} \in \mathbf{M}\}$. According to the results of Section 6.4.2, $\mathrm{orb_n}(\mathcal{H}_p, \sigma_{\mathbf{M}})$ is the best approximation element for $\mathrm{orb}_n(\mathcal{H}_p, f) \in D(\mathcal{H}_p^n)$ in the orthogonal complemented subspace to the subspace $\{y \in D(\mathcal{H}_p^n) : \langle y, H_{\mathbf{k}}\rangle_n = 0, \mathbf{k} \in \mathbf{M}\}$ with respect to the hilbertian norm $\|\cdot\|_n$ and has the form

$$\mathrm{orb}_n(\mathcal{H}_p, \sigma_{\mathbf{M}}) = \sum_{\mathbf{k}\in\mathbf{M}} \frac{\langle f, H_{\mathbf{k}}\rangle_n}{\langle H_{\mathbf{k}}, H_{\mathbf{k}}\rangle_n}\ \mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}). \tag{6.5.9}$$

It is clear that the generating element of this orbit is

$$\sigma_{\mathbf{M}} = \sum_{\mathbf{k}\in\mathbf{M}} \frac{(f, H_{\mathbf{k}}}{(H_{\mathbf{k}}, H_{\mathbf{k}})} H_{\mathbf{k}}.$$

We have

$$\begin{aligned}\langle f, H_{\mathbf{k}}\rangle_n &= (f, H_{\mathbf{k}}) + (\mathcal{H}_p f, \mathcal{H}_p H_{\mathbf{k}}) + \cdots + (\mathcal{H}_p^n f, \mathcal{H}_p^n H_{\mathbf{k}})\\ &= (f, H_{\mathbf{k}}) + (f, \mathcal{H}_p^2 H_{\mathbf{k}}) + \cdots + (f, \mathcal{H}_p^{2n} H_{\mathbf{k}})\\ &= (f, H_{\mathbf{k}}) + (f, (2|\mathbf{k}|+p)^2 H_{\mathbf{k}}) + \cdots + (f, (2|\mathbf{k}|+p)^{2n} H_{\mathbf{k}})\\ &= (1 + (2|\mathbf{k}|+p)^2 + \cdots + (2|\mathbf{k}|+p)^{2n})(f, H_{\mathbf{k}}).\end{aligned}$$

Since $\langle H_{\mathbf{k}}, H_{\mathbf{k}}\rangle_n = (1 + (2\mathbf{k}) + p)^2 + \cdots + (2\mathbf{k}) + p)^{2n})$, it follows from here that

$$\frac{\langle f, H_{\mathbf{k}}\rangle_n}{\langle H_{\mathbf{k}}, H_{\mathbf{k}}\rangle_n} = (f, H_{\mathbf{k}})$$

and we may rewrite (6.5.9) in the form

$$\mathrm{orb}_n(\mathcal{H}_p, \sigma_{\mathbf{M}}) = \sum_{\mathbf{k}\in\mathbf{M}} (f, H_{\mathbf{k}})\mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}).$$

The left inverse $S_n : \mathcal{H}_p^{n+1} \to \mathcal{H}_p^{n+1}$ to the operator $\mathcal{H}_{pn}$, i.e. the solution operator of the equation (6.5.6), is defined by $S_n(\mathrm{orb}_n(\mathcal{H}_p, \mathcal{H}_p x) = \mathrm{orb}_n(\mathcal{H}_p, x)$ and is the positive operator on $\mathrm{Im}\mathcal{H}_{pn}$. It is clear that $S_n(\mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{M}})) = \sum_{\mathbf{k}\in\mathbf{M}}(2|\mathbf{k}| + 1)^{-1}\mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{k}})$ and we obtain

$$\begin{aligned}\mathrm{orb}_n(\mathcal{H}_p, u_{\mathbf{M}}) &= S_n(\mathrm{orb}_n(\mathcal{H}_p, \sigma_{\mathbf{M}})) = S_n \sum_{\mathbf{k}\in\mathbf{M}} (f, H_{\mathbf{k}})\mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{k}})\\ &= \sum_{\mathbf{k}\in\mathbf{M}} \frac{1}{2|\mathbf{k}|+p}(f, H_{\mathbf{k}})\mathrm{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}),\end{aligned} \tag{6.5.10}$$

and its generating element of is

$$u_{\mathbf{M}} = \sum_{\mathbf{k}\in\mathbf{M}} \frac{1}{2|\mathbf{k}|+p}(f, H_{\mathbf{k}})H_{\mathbf{k}}. \tag{6.5.11}$$

This means that $U(f)=\operatorname{orb}_n(\mathcal{H}_p,u_{\mathbf{M}})$ is a linear spline algorithm for the information $[\langle f,H_{\mathbf{k}}\rangle_n,\ \mathbf{k}\in\mathbf{M}]$. In a completely analogous way to the one-dimensional case [1] (see Section 1.4), it is proved that the linear spline algorithm (6.5.10) is also a central algorithm in the space $D(\mathcal{H}_p^n)$.

Let us now consider the problem of the convergence of this algorithm. Consider a series

$$\sum_{\mathbf{n}\in\mathbf{Q}^p} c_{\mathbf{n}}H_{\mathbf{n}} \tag{6.5.12}$$

with the complex coefficients $c_{\mathbf{n}}$. They say that a series (6.5.12) converges unconditionally in square mean to a function $f\in L^2(\mathbb{R}^p)$ if and only if, given any number $\varepsilon>0$, there is a finite set $\mathbf{M}_0\subset\mathbf{Q}^p$ such that $\|f-\sum\limits_{\mathbf{n}\in\mathbf{M}} c_{\mathbf{n}}H_{\mathbf{n}}\|<\varepsilon$ for every finite set $\mathbf{M}$ such that $\mathbf{M}_0\subset\mathbf{M}\subset\mathbf{Q}^p$. In ([10], Chapter 3), it is proved that the series (6.5.12) converges unconditionally in square to a function $f\in L^2(\mathbb{R}^p)$ if and only if $\sum\limits_{\mathbf{n}\in\mathbf{Q}^p}|c_{\mathbf{n}}|^2<\infty$, moreover, $c_{\mathbf{n}}=(f,H_{\mathbf{n}})$, and $\|f\|^2_{L^2(\mathbb{R}^p)}=\sum\limits_{\mathbf{n}\in\mathbf{Q}^p}|c_{\mathbf{n}}|^2$.

Now, we consider the problem of the convergence of the sequence $\operatorname{orb}_n(\mathcal{H}_p,u_{\mathbf{M}})$, where $\mathbf{M}\subset\mathbf{Q}^p$. If for the series (6.5.12) unconditionally converging in the mean square to the function $f\in L^2(\mathbb{R}^p)$, we construct its partial sum $f_{\mathbf{M}}=\sum\limits_{\mathbf{n}\in\mathbf{M}} c_{\mathbf{n}}H_{\mathbf{n}}$, then according to the definition given above, this sum can be called unconditionally converging in the mean square to f and we write this in the form $\sum\limits_{\mathbf{n}\in\mathbf{M}} c_{\mathbf{n}}H_{\mathbf{n}}$ converges to f. Thus, the unconditionally in the mean square convergence of a series (6.5.12) to a function f is equivalent to the unconditionally in the mean square convergence of its partial sums $f_{\mathbf{M}}$ to the f.

Let us prove that the defined by (6.5.10) sequence $\operatorname{orb}_n(\mathcal{H}_p,u_{\mathbf{M}})$ unconditionally in the mean square converges to $\operatorname{orb}_n(\mathcal{H}_p,f)$, if $\mathcal{H}_p^j u_{\mathbf{M}}\Rightarrow\mathcal{H}_p^{j-1}f$ for each $1\le j\le n+1$ and $u_{\mathbf{M}}$ converges in $L^2(\mathbb{R}^p)$ to u^*. According to (6.5.11), we have

$$\begin{aligned}
\mathcal{H}_p^j u_{\mathbf{M}} &= \mathcal{H}_p^j\Big(\sum_{\mathbf{k}\in\mathbf{M}}\frac{1}{2|\mathbf{k}|+p}(f,H_{\mathbf{k}})H_{\mathbf{k}}\Big)\\
&= \sum_{\mathbf{k}\in\mathbf{M}}\frac{1}{2|\mathbf{k}|+p}(f,H_{\mathbf{k}})\mathcal{H}_p^j H_{\mathbf{k}}\\
&= \sum_{\mathbf{k}\in\mathbf{M}}(f,H_{\mathbf{k}})\mathcal{H}_p^{j-1}H_{\mathbf{k}}=\sum_{\mathbf{k}\in\mathbf{M}}(\mathcal{H}_p^{j-1}f,H_{\mathbf{k}})H_{\mathbf{k}}=\mathcal{H}_p^{j-1}f.
\end{aligned}$$

Because of $2|\mathbf{k}|+p\ge 1$, for the coefficients $c'_{\mathbf{n}}$ of the Fourier series

$$\sum_{\mathbf{k}\in\mathbf{Q}^p}\frac{1}{2|\mathbf{k}|+p}(f,H_{\mathbf{k}})H_{\mathbf{k}}$$

we have that $c'_{\mathbf{k}} \leq c_{\mathbf{k}}(f) = (f, H_{\mathbf{k}})$. Therefore, $\sum_{\mathbf{k}\in\mathbf{Q}^p} (c'_{\mathbf{k}})^2 < \infty$ and the series $\sum_{\mathbf{k}\in\mathbf{Q}^p} \frac{1}{2|\mathbf{k}|+p}(f, H_{\mathbf{k}})H_{\mathbf{k}}$ converges unconditionally in the mean square to some $u^* \in L^2(\mathbb{R}^p)$. In the case when the equation $\mathcal{H}_p u = f$ has a unique solution $u_0 \in L^2(\mathbb{R}^p)$, we have

$$\sum_{\mathbf{k}\in\mathbf{Q}^p} (u_0, H_{\mathbf{k}})H_{\mathbf{k}} = u_0$$

and

$$\sum_{\mathbf{n}\in\mathbf{Q}^p} (u_0, H_{\mathbf{n}})H_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathbf{Q}^p} \frac{1}{2|\mathbf{n}|+p}(u_0, \mathcal{H}_p H_{\mathbf{n}})H_{\mathbf{n}}$$
$$= \sum_{\mathbf{n}\in\mathbf{Q}^p} \frac{1}{2|\mathbf{n}|+p}(\mathcal{H}_p u_0, H_{\mathbf{n}})H_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathbf{Q}^p} \frac{1}{2|\mathbf{n}|+p}(f, H_{\mathbf{n}})H_{\mathbf{n}} = u^*.$$

That is, $u_0 = u^*$. We may say that the following theorem is true.

Theorem 6.5.1. *Let* $f \in D(\mathcal{H}_p^n)$ *and a nonadaptive information* $I(f) = \{\langle f, H_{\mathbf{k}}\rangle_n, \mathbf{k} \in \mathbf{M}\}$ *of cardinality* $\mathbf{M} \subset \mathbf{Q}^p$ *be given. Then the algorithm*

$$\operatorname{orb}_n(\mathcal{H}_p, u_{\mathbf{M}}) = \sum_{\mathbf{k}\in\mathbf{M}} \frac{1}{2|\mathbf{k}|+p}(f, H_{\mathbf{k}}) \operatorname{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}) \tag{6.5.13}$$

is linear spline central algorithm for an approximate solution of n*-orbital equation* (6.5.6) *in the space of* n*-orbits* $D(\mathcal{H}_p^n)$. *The sequence* $\{\operatorname{orb}_n(\mathcal{H}_p, \mathcal{H}_p u_{\mathbf{M}}\}$ *converges unconditionally in the mean square to* $\operatorname{orb}_n(\mathcal{H}_p, f)$ *in the space* $D(\mathcal{H}_p^n)$ *and* $\{u_{\mathbf{M}}\}$ *converges unconditionally in the mean square to some* $u^* \in L^2(\mathbb{R}^p)$.

If f is such function from $D(\mathcal{H}_p^n)$ that the functional (u, f) is bounded on the energetic space $H_{\mathcal{H}_{pn}}$, then $(u, f) = [u, u_0]$, where $[\cdot,\cdot]_{H_{\mathcal{H}_{pn}}}$ is inner product on $H_{\mathcal{H}_p}$. Since the operator $\mathcal{H}_p$ is positive on H, according to ([**?**], Chapter 1, p. 28), the following two cases are possible: if $u_0 \in H$, then $u_0 \in H_{\mathcal{H}_p}$ and $\mathcal{H}_p u_0 = f$; if $u_0 \in H$, it can be called a generalized solution of the equation $\mathcal{H}_p u = f$.

Theorem 6.5.2. *In the Fréchet space of all orbits* $D(\mathcal{H}_p^\infty) = S(\mathbb{R}^p)$ *the algorithm*

$$\operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}}) = \sum_{\mathbf{k}\in\mathbf{M}} \frac{1}{2|\mathbf{k}|+p}(f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}}) \tag{6.5.14}$$

is a linear central spline algorithm in the space $D(\mathcal{H}_p^\infty)$. *The equation* (6.5.8) *has a unique solution generated by some* $u_0 \in L^2(\mathbb{R}^{\prime})$, *the sequence* (6.5.14) *converges unconditionally to* $\operatorname{orb}(\mathcal{H}_p, u_0)$ *and the following estimates hold true:*

a) *for every* n,

$$\|\operatorname{orb}(\mathcal{H}_p, u_0) - \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})\|_n \leq \|\operatorname{orb}(\mathcal{H}_p, f) - \operatorname{orb}(\mathcal{H}_p, \sigma_{\mathbf{M}})\|_n;$$

b) *for every* $n \in \mathbb{N}_0$, *there exists* $\mathbb{N}_0 = \mathbb{N}_0(n)$ *such that for every* $\mathbf{M} \supset \mathbb{N}_0$,

$$\|\operatorname{orb}(\mathcal{H}_p, u_0) - \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})\|_n \leq |\operatorname{orb}(\mathcal{H}_p, f) - \operatorname{orb}(\mathcal{H}_p, \sigma_{\mathbf{M}})|;$$

c) *for every* $\mathbf{M} \in \mathbf{Q}^p$,

$$|\operatorname{orb}(\mathcal{H}_p, u_0) - \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})| \leq |\operatorname{orb}(\mathcal{H}_p, f) - \operatorname{orb}(\mathcal{H}_p, \sigma_{\mathbf{M}})|.$$

Proof. The proof is similar to the proof of Theorem 6.4.1 and we present it only for the case a). It is easy to verify that the series $\sum_{\mathbf{k}\in\mathbf{Q}^p} \lambda_{\mathbf{k}}^{-1}(f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}})$, converges unconditionally to $\operatorname{orb}_n(\mathcal{H}_p, u_{\mathbf{0}})$. Since the equation (6.5.7) has a unique solution generated by some $u_0 \in L^2(\mathbb{R}^{\prime})$, then

$$\operatorname{orb}(\mathcal{H}_p, u_0) = S_p(\operatorname{orb}(\mathcal{H}_p, f) = \sum_{\mathbf{k}\in\mathbf{Q}^p} \lambda_{\mathbf{k}}^{-1}(f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}}),$$

where S_p is the inverse for the operator $\mathcal{H}_p$ in the space $D(\mathcal{H}_p^\infty)$. Thus, the sequence $\{\operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})\}$ converges to the exact solution in $D(\mathcal{H}_p^\infty)$.

We have that if $f \in D(\mathcal{H}_p^\infty)$, then

$$\operatorname{orb}(\mathcal{H}_p, f) = \sum_{\mathbf{k}\in\mathbf{Q}^p} (f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}})$$

and

$$\begin{aligned}\mathcal{H}_{pn} \operatorname{orb}(\mathcal{H}_p, f) &= \operatorname{orb}_n(\mathcal{H}_p, \mathcal{H}_p f) = \sum_{\mathbf{k}\in\mathbf{Q}^p} (\mathcal{H}_p f, H_{\mathbf{k}}) \operatorname{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}) \\ &= \sum_{\mathbf{k}\in\mathbf{Q}^p} (f, \mathcal{H}_p H_{\mathbf{k}}) \operatorname{orb}_n(\mathcal{H}_p, H_{\mathbf{k}}) = \sum_{\mathbf{k}\in\mathbf{Q}^p} \lambda_{\mathbf{k}}(f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}}),\end{aligned}$$

where $\lambda_{\mathbf{k}} = 2|k| + p$. Then, for any $n \in \mathbb{N}_0$,

$$\begin{aligned}&\langle \mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, f), \operatorname{orb}(\mathcal{H}_p, f)\rangle_n \\ &\quad= \sum_{\mathbf{k}\in\mathbf{Q}^p} \lambda_{\mathbf{k}} \langle (f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}}), \sum_{\mathbf{k}\in\mathbf{Q}^p} \lambda_{\mathbf{k}} \langle (f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}})\rangle_n \\ &\quad\geq \lambda_0 \langle \sum_{\mathbf{k}\in\mathbf{Q}^p} (f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}}), \sum_{\mathbf{k}\in\mathbf{Q}^p} (f, H_{\mathbf{k}}) \operatorname{orb}(\mathcal{H}_p, H_{\mathbf{k}})\rangle_n \\ &\quad= \lambda_0 \langle \operatorname{orb}(\mathcal{H}_p, f), \operatorname{orb}(\mathcal{H}_p, f)\rangle_n. \qquad (6.5.15)\end{aligned}$$

Since

$$\langle \mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, f), \operatorname{orb}(\mathcal{H}_p, f)\rangle_n \leq \|\mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, f)\|_n \|\operatorname{orb}(\mathcal{H}_p, f)\|_n, \quad n \in \mathbb{N}_0,$$

(6.5.15) implies that

$$\|\operatorname{orb}(\mathcal{H}_p, u_0)\|_n \leq \|\mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, f)\|_n,$$

i.e.

$$\begin{aligned}\|\operatorname{orb}(\mathcal{H}_p, u_0) - \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})\|_n &\leq \|\mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, u_0) - \mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}})\|_n \\ &= \|\mathcal{H}_p^\infty S_p \operatorname{orb}(\mathcal{H}_p, f) - \mathcal{H}_p^\infty \operatorname{orb}(\mathcal{H}_p, u_{\mathbf{M}}) \\ &= \|\operatorname{orb}(\mathcal{H}_p, f) - \operatorname{orb}(\mathcal{H}_p, \sigma_{\mathbf{M}})\|_n\end{aligned}$$

and the case a) of Theorem 6.5.2 is proved. □

6.6 Syllabus of the course: Approximate calculation in quantum mechanics (QM)

Title of the Course: Course Identification Code: PHS61508E2-LSB

Academic Degree of Higher Education: Master's degree

Teaching Language: English

Course Author/Authors:

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This course is teaching since 2022 to Master's students at the Nuclear Engineering specialty of the Faculty of Informatics and Control Systems Georgian Technical University. The syllabus is based on a monograph "Central Spline Algorithms in the Hilbert and Fréchet Spaces of Orbits".

Course Description:
Explanation of the existence of stable interdependence of particle complexes (atoms, molecules, nuclei, bosons) considered in QM and description of energy processes

associated with these particle complexes. For the mathematical solution of these tasks, the study of the methods of approximate solution of the differential and integral equations containing the discrete-spectrum operator of the system energy and related operators and familiarization with the created software.

Course Prerequisites:
Computing and engineering geometry modeling in high energy experiments

Learning Outcomes:

- *Knowledge and skills acquired as a result of studying the subject*
  1. realizes the need and necessity of using approximate methods in the tasks of simulation and visualization of geometric models and in high-energy physical experiments;
  2. Analyzes approximate methods and differential equation apparatus for solving equations containing hamiltonian of QHO and Schrödinger operator and adapts their software packages to the needs of users' tasks;
  3. programs approximate methods and differential equations machine algorithms and evaluates the adequacy of computer modeling of relevant processes;
- *Methods of Achieving Learning Outcomes (Teaching and Learning):*
  Lecture; Seminar (team working); Laboratory; Consultation; Independent work
- *Activities of the teaching-learning methods:*

The method of synthesis involves grouping separate issues to form a single whole. This method helps to develop the ability to see the problem as a whole.

Laboratory work is more prominent and gives an opportunity to perceive the event or process. In the laboratory, the student studies how to conduct the experiment. During laboratory studies, the student must acquire to organize, regulate and work on the device. The skills obtained in experimental training labs allow to understand theoretical material delivered on the lectures. It implies the following types of actions: setting up the tests, showing video material, as well as the material of dynamic nature, and so forth.

Analysis helps us to divide the study material into constituent parts. This will simplify the detailed coverage of individual issues within a difficult problem.

Verbal or orally transmitted. Narration, talking and so forth belong to this activity. In this process the teacher orally transmittes and explaines study material and the students actively perceive and learn it through listening, remembering and thinking.

Action-oriented training requires active involvement of the teacher and student in the teaching process, where the practical interpretation of theoretical material is of special significance.

Explanation is based on the discussion on the issue. The teacher gives a concrete example from the material, which is discussed in detail within the given topic.

Credits: 5

Course Schedule in Accordance to the Students' Weekly Workload (hrs.)

Lecture:	15
Seminar (work in the group):	
Practical classes:	15
Laboratory:	
Course work/project:	15
Practice:	
Mid-semester/final exam:	1/2
Independent work:	77

Lecture
Title of the theme and its content

1. Introduction. A brief history of the beginnings of quantum mechanics: Planck's constant, Bohr's planetary model, Einstein's photo effect, de Broglie's wave mechanics, Heisenberg's uncertainty principle. Mathematical model: Schrödinger's equation, Dirac's equation, von Neumann's mathematical apparatus of quantum mechanics.
2. Quantum Hilbert space of states of quantum-mechanical systems, observable quantities in classical physics: energy, position and momentum. Their quantization and observables in QM - self-adjoint unbounded operators of energy, position and momentum in quantum Hilbert space.
3. Self-adjoint operators with discrete spectrum in quantum Hilbert space, hamiltonian of harmonic oscillator, one-dimensional Schrödinger equation. Hermite functions - basis in quantum Hilbert space. The principle of superposition in quantum mechanics, Schrödinger's cat.
4. Best approximation problems in Banach and Hilbert spaces, central and optimal algorithms for linear problems with absolutely convex set of problem elements in Banach space.
5. Interpolation and spline algorithms for non-adaptive information, existence of Chebyshev center.
6. Ritz method in energetic Hilbert space, least squares method and their centrality, central spline algorithms for the approximate solution of equations containing Schrödinger's one- dimensional discrete spectrum differential operator, Tricom, Laplace–Beltram.

7. James and Bishop-Phelps theorems, existence and linearity conditions of spline algorithm.
8. Linear, central spline algorithm in orbital Hilbert space $D(A^n)$ for the equation $Au = f$. Linear, spline algorithm in orbital Hilbert space $D(K^{-n})$ for the equation $Ku = f$. Applications to first- order integral equations for the approximate solution of the equation involving the inverse of a QHO.
9. A linear, spline algorithm for the equation $Au = f$ with an operator A that admits a SVD in the space $D((A^*A)^{-n})$. Using for the equation $Ru = f$ in orbital Hilbert space $D((R^*R)^{-n})$, where R is the Radon transform in CT.
10. Strictly distinguished Fréchet spaces and strict Fréchet–Hilbert Spaces. Strict Fréchet–Hilbert space $L^2_{loc}(R)$ of locally integrable functions of quantum-mechanical systems. Its geometric and topological properties. Best approximation theory issues in quantum Fréchet–Hilbert Spaces. Canonical commutative relations for orbital operators, Heisenberg's uncertainty principle.
11. Optimization problems of convex and quasi-convex functionals and their applications. A functional (metric) used to reduce an infinite-coordinate computing process to a finite- coordinate computing process. Creating software and performing calculations according to the constructed algorithm.
12. Continuity of self-adjoint operators of position, moment and energy of observable quantities in the space of generalized functions. Consideration of these operators in the Schwartz space of rapidly decreasing functions and in the infinite order Sobolev space. Solving the Schrödinger equation using the finite difference method, high-order differential equations in Fréchet–Hilbert spaces of all orbits.
13. Central and optimal algorithms for linear problems with sequences of absolutely convex sets of problem elements. Existence of spline and spline algorithm in Fréchet spaces of all orbits. The existence of a spline for non-adaptive information of one cardinality. Generalizations of James and Bishop–Phelps theorems.
   Construction of linear spline and central algorithm for approximate solution of Schrödinger equation in orbital spaces. Creating software and performing calculations based on spline algorithms.
14. Generalization of the Ritz method for the approximate solution of the equation containing the QHO in the space of all orbits. Construction of a linear central spline algorithm for the approximate solution of the equation containing the QHO in the Fréchet–Hilbert space of all orbits. Construction of a spline algorithm for an ill-posed problem. their stability, performing calculations according to built-spline algorithms.

15. Continuation of self-adjoint position, momentum and energy operators of observables in strict Fréchet–Hilbert space. Solving the equations containing discrete spectrum Sturm- Liouville, Beltrame, and other operators in the space of all orbits. Application of the generalization of the least squares method for the approximate solution of the equation containing the QHO in the space of all orbits.

Practical classes
Title of the theme and its content

1. Mathematical model of quantum mechanics: Schrödinger's equation, Dirac's equation, von Neumann's mathematical apparatus of QM. Finite-dimensional space and its use in the process of creating quantum computers.
2. Matrix calculations. Matrices and vectors. Linear space and algebra of matrices.
3. Best approximations, Ritz's method for approximate solution of an equation containing a symmetric operator.
4. Solving a system of linear equations: simple iterations method, Monte-Carlo method. The method of least squares for the approximate solution of the equation containing the symmetric operator.
5. Linear interpolation. Existence of interpolation spline, polynomial interpolation. Lagrange interpolation polynomial. Spline construction for Birkhoff data.
6. Polynomial interpolation. Best approximations with Hermite polynomials in Hilbert space of states of quantum mechanical systems. Interpolation spline and spline algorithm.
7. Construction of second order parabolic spline. Creating software and performing calculations based on spline algorithms. The principle of superposition in quantum mechanics, Schrödinger's cat.
8. Linear, central spline algorithm in orbital Hilbert space $D(A^n)$, linear, spline algorithm in orbital Hilbert Space $D(K^{-n})$, applications to first order integral equations, approximate solution of equations containing the inverse of QHO.
9. Best approximation theory issues in quantum Fréchet–Hilbert spaces. A central spline algorithm in the Fréchet–Hilbert space of all orbits $D(A^\infty)$. Creating software and performing calculations according to the constructed algorithm.

10. Linear spline and central algorithms, Chebyshev center concept in strict Fréchet–Hilbert spaces. Error estimation. Convex optimization problems
11. A functional (metric) used to reduce an infinite-coordinate computing process to a finite-coordinate computing process. Getting to know the software and performing calculations according to the constructed algorithm.
12. Solving the Schrödinger equation using the difference method. Higher order differential equations in orbital spaces. Creating software and performing calculations based on the constructed spline algorithm.
13. Central and optimal algorithms for linear second-order differential equations with sequences of absolutely convex sets of problem elements. Existence of generalized spline and spline algorithm in Fréchet–Hilbert spaces of all orbits. Existence of a spline for nonadaptive information of one cardinality. Generalizations of James and Bishop-Phelps theorems.
14. Construction of a spline algorithm for the integral equation for an ill-posed problem of scanner of CT. Creating software, its stability, performing calculations according to the constructed spline algorithm.
15. Application of the generalization of the least squares method for the approximate solution of the equation containing the QHO in the space of all orbits.

Laboratory
Title and content of topics

1. Finite-dimensional space and the coordinates of its vectors, the angle between them, the lengths of the vectors. Cartesian product and some properties.
2. An infinite-dimensional quantum Hilbert space and its finite-dimensional subspaces. A quantum Hilbert space basis with Hermite functions,
3. Gaussian elimination method for solving systems of linear equations, systems of linear equations with diagonal and near-diagonal matrices. Error calculation.
4. Solving a system of linear equations: simple iterations method, Monte-Carlo method. The method of least squares for the approximate solution of the equation containing the symmetric operator.
5. Linear interpolation. Existence of interpolation spline, polynomial interpolation. Construction of Lagrange interpolation polynomial Spline for Birkhoff data.

6. Creating software in the computing program "Mathematics" and performing calculations based on spline algorithms in quantum Hilbert space.
7. Constructing a parabolic spline. Interpolation spline, superposition principle in quantum mechanics, Schrödinger's cat.
8. Approximate solution of equations involving the inverse of a QHO and relation between solutions of direct and inverse equations.
9. Best approximation theory issues in quantum Fréchet–Hilbert spaces. A central spline algorithm in the Fréchet–Hilbert space for all orbits $D(A^{\infty})$. Creating software and performing calculations according to built algorithms.
10. Linear spline and central algorithms. The notion of Chebyshev center in strict Fréchet- Hilbert spaces. Error estimation. Convex optimization problems.
11. A constructed by us functional (metric) used to reduce an infinite-coordinate computing process to a finite- coordinate computing process. Creating software and performing calculations according to the constructed algorithm.
12. Solving the Schrödinger equation using the difference method. Higher order differential equations in orbital spaces. Modeling task, creating software and performing calculations based on built-in spline algorithms.
13. Construction of linear spline and central algorithm for approximate solution of Schrödinger equation in orbital spaces. Software and calculations based on spline algorithm.
14. Construction of a spline algorithm for the CT scanner ill-posed problem. Its stability, performing calculations based on built of spline algorithms.
15. Application of the generalization of the least squares method for the approximate solution of the equation containing the QHO in the space of all orbits.

Student Knowledge Assessment System,
Grading system is based on a 100-point scale.

Positive grades:

- (A) - Excellent - grades between 91-100 points;
- (B) – Very good - grades between 81-90 points
- (C) - Good - grades between 71-80 points
- (D) - Satisfactory - grades between 61-70 points
- (E) - Pass - the rating of 51-60 points

Negative grades:

- (FX) - Did not pass - grades between 41-50 points, which means that the student is required to work more to pass and is given the right, after independent work, to take one extra exam;
- (F) – Failed - 40 points and less, which means that the work carried out by the student did not bring any results and he/she has to learn the subject from the beginning.

Assessment Forms, Methods and Criteria
Ongoing activity:
Maximal total assessment of the current activity is 30 points. Assessment is based on homework, quizzes, which should be prepared independently by a student and presented in a written form.

Assessments of the current activity for the home works and quizzes are distributed as follows:

Home works are estimated by 18 points. During a semester student has to present the written home works 3 times. Home works include 3 topics with maximal assessment of 2 points each, summing up in total to 6 points.

Quizzes are estimated by 12 points. During a semester student has to write 3 quizzes. The quiz includes 2 topics with maximal assessment of 2 points each, summing up in total to 4 points. The quiz is conducted at the seminars.

2 p. - Task is executed completely, precisely and exhaustively. The special terminology is proper. The student knows material and the corresponding literature very well. There are no mistakes made. Argumentation is on a high level.

1 p. - Execution of the task is incomplete. Student knows the curriculum material but there are some drawbacks. Argumentation is fragmentary.

0 p. - The execution is not adequate to the task, or is not given at all.

Mid-semester exam:

Maximal total number of points for a mid-semester exam is 30. Mid-semester exam is conducted once in a semester and is a necessary component for the mid-term assessment.

Mid-semester exam is conducted in a written form using open-ended questions. Student should be able to present the knowledge of studied topics.

Exam list contains 6 topics with maximal assessment of 5 points each, summing up in total to 30 points.

5 p. - The task is executed completely, precisely and exhaustively. The special terminology is proper. The student knows material and the corresponding literature very well. There are no mistakes made. Argumentation is on a high level.

4 p. - The execution of the task is almost complete, but some details are missing and some inessential errors are observed. Student knows material and is well aware of the corresponding literature. There are no mistakes made. Argumentation is good.

3 p. - The execution of the task is incomplete. Student knows the curriculum material but some mistakes are observed.

2 p. - The execution of the task is incomplete or wrong. Student knows the curriculum material but some essential drawbacks are observed. Argumentation is fragmentary.

1 p. - Only some fragments of the topic are presented. Argumentation is not correct. The absence of the deep knowledge is observed.

0 p. - The execution is not adequate to the task, or is not given at all.

Final/additional exam:

Maximal total point for the final/additional exam is 40.

A student who could not receive at least 30 points in total in the ongoing activity and the mid-semester exam will not be admitted to the final/additional exam. Minimum positive assessment of the final/additional exam is 21 points.

The final/additional exam is conducted in the written form. Student should be able to present the knowledge of the studied topics. Examination list contains 8 topics, with maximal assessment of 5 points each, summing up in total to 40 points.

5 p. - The task is executed completely, precisely and exhaustively. The special terminology is proper. The student knows material and the corresponding literature very well. There are no mistakes made. Argumentation is on a high level.

4 p. - The execution of the task is almost complete, but some details are missing and some inessential errors are observed. Student knows material and is well aware of the corresponding literature. There are no mistakes made. Argumentation is good.

3 p. – Initial argumentation is basically correct, but further steps are incorrect or incomplete. The answer is incomplete. Student knows the curriculum material but some mistakes are observed.

2 p. - The execution of the task is incomplete or wrong. Student knows the curriculum material but some essential drawbacks are observed. Argumentation is fragmentary.

1 p. - Only some fragments of the topic are presented. Argumentation is not correct. The absence of the deep knowledge is observed.

0 p. - The execution is not adequate to the task, or is not given at all.

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